GEOMETRY

A. GROTHENDIECK

This text has been transcribed and edited by Mateo Carmona with the collaboration of Tim Hosgood. Remarks, comments, and corrections are welcome.

https://agrothendieck.github.io/

CONTENTS

A GENERAL THEORY OF FIBRE SPACES WITH STRUCTURE SHEAF

Introduction

When one tries to state in a general algebraic formalism the various notions of fibre space: general fibre spaces (without structure group, and maybe not even locally trivial); or fibre bundle with topological structure group G as expounded in the book of Steenrod ([]); or the "differeniable" and "analytic" (real or complex) variants of theses notions; or the notions of algebraic fibre spaces (over an abstract field k) - one is led in a natural way to the notion of fibre space with a structure sheaf $\mathbf{G}$. This point of view is also suggested a priori by the possibility, now classical, to interpret the (for instance "topological") classes of fibre bundles on a space X, with *abelian* structure group G, as the elements of the first cohomology group of X with coefficients in the sheaf $\mathbf{G}$ of germs of continuous maps of X into G; the word "continuous" being replaced by "analytic" respectively "regular" if G is supposed an analytic respectively an algebraic group (the space X being of course accordingly an analytic or algebraic variety). The use of cohomological methods in this connection have proved quite useful, and it has become natural, at least as a matter of notation, even when G is not abelian, to denote by $\mathrm{H}^1(X, \mathbf{G})$ the set of classes of fibre spaces on X with structure sheaf $\mathbf{G}$, $\mathbf{G}$ being as above a sheaf of germs of maps (continuous, or differentiable, or analytic, or algebraic as the case may be) of X into G. Here we develop systematically the notion of fibre space with structure sheaf $\mathbf{G}$, where $\mathbf{G}$ is any sheaf of (not necessarily abelian) groups, and of the first cohomology set $H^1(X, \mathbf{G})$ of X with coefficients in $\mathbf{G}$. The first four chapters contain merely the first definitions concerning general fibre spaces, sheaves, fibre spaces with composition law (including sheaves of groups) and fibre spaces with structure sheaf. The functor aspect of the notions dealt with has been stressed throughout, and as it now appears should have been stressed even more. As the proofs of most of the facts stated reduce of course to straightforward verifications, they are only sketched or even omitted, the important point being merely a consistent order in the statement of the main facts. In the las chapter, we define the cohomology set $H^1(X, \mathbf{G})$ of X with coefficients in the sheaf of groups $\mathbf{G}$,

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I. General fibre spaces

Unless otherwise stated, none of the spaces to occur in this report have to be supposed separated.

1.1 Notion of fibre space

Definition **1.1.1**. — *A* fibre space *over a space X is a triple (X, E, p) of the space X, a space E and a continuous map p of E into X.*

We do not require p to be onto, still less to b open, and if p is onto, we do not require the topology of X to be the quotient topology of E by the map p. For abbreviation, the fibre space (X, E, p) will often be denoted by E only, it being understood that E is provided with the supplementary structure consisting of a continuous map p of E into the space X. X is called the *base space* of the fibre space, p the *projection*, and for any $x \in X$, the subspace $p^{-1}(x)$ of E (which is closed if $\{x\}$ is closed) is the *fibre* of x (in E).

Given two fibre spaces (X, E, p) and (X', E', p'), a *homomorphism* of the first into the second is a pair of continuous maps $f : X \longrightarrow X'$ and $g : E \longrightarrow E'$, such that $p' g = f p$, i.e. commutativity holds in the diagram

$$\begin{array}{ccc} E & \xrightarrow{g} & E' \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle p'} \\ X & \xrightarrow[f]{} & X' \end{array}$$

Then g maps fibres into fibres (but not necessarily *onto*!); furthermore, if p is surjective, then f is uniquely determined by g. The continuous map f of X into X' being given, g will be called also a f-homomorphism of E into E'. If, moreover, E'' is a fibre space over X', f' a continuous map $X' \longrightarrow X''$ and $g' : E' \longrightarrow E''$ a f'-homomorphism, then $g'g$ is a $f'f$-homomorphism. If f is the identity map of X onto X, we say also X-homomorphism instead of f-homomorphism. If we speak of homomorphisms of fibre spaces over X, without further comment, we will always mean X-homomorphisms.

The notion of *isomorphism* of a fibre space (X,E,p) onto a fibre space (X',E',p') is clear: it is a homomorphism (f,g) of the first into the second, such that f and g are onto-homeomorphisms.

1.2 Inverse image of a fibre space, inverse homomorphisms

Let (X,E,p) be a fibre space over the space X, and let f be a continuous map of a space X' into X. Then the *inverse image* of the fibre space E by f is a fibre space E' over X'. E' is defined as the subspace of $X' \times E$ of points (x',y) such that $fx' = py$, the projection p' of E' into the base X' being given by $p'(x',y) = x'$. The map $g(x',y) = y$ of E' into E is then an f-homomorphism, inducing for each $x' \in X'$ a *homeomorphism* of the fibre of E' over x' *onto* the fibre of E over $f'x'$.

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1.3 Subspace, quotient, product

Let (X,E,p) be a fibre space, E' any subspace of E, then the restriction p' of p to E', defines E'

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1..4 Trivial and locally trivial fibre spaces

Let X and F be two spaces, E the product space, the projection of the product on X defines E as a fibre space over X, called the *trivial fibre space over X with fibre F*.

All fibres are canonically homeomorphic with F.

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1.5 Definition of fibre spaces by coordinate transformations

Let X be a space, (U_i) a covering of X, for each

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1.6 The case of locally trivial fibre spaces

The method of the preceding section for constructing fibre spaces over X will be used mainly in the case where we are given a fibre space over T over X, and where, given an open covering (U_i) of X, we consider the fibre spaces

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1.7 Sections of fibre spaces

Definition **1.7.1**. — *Let* (X, E, p) *b a fibre space; a* section *of this fibre space (or, by pleonasm, a section of* E *over* X*) is a map* x *of* X *into* E *such that* ps *is the the identity map of* X*. The set of* continuous *sections of* E *is noted* $\mathrm{H}^0(X, E)$.

It amounts to the same to say that s is a function the value of which at each $x \in X$ is in the fibre of x in E (which depends on x!).

The existence of a section implies of course that p is onto, and conversely if we do not require continuity. However, we are primarily interested in continuous sections. A *section of* E *over a subset* Y of X is by definition a section of $E|Y$. If Y is open, we write $\mathrm{H}^0(Y, E)$ for the set $\mathrm{H}^0(Y, E|Y)$ of all continuous sections of E over Y.

$\mathrm{H}^0(X, E)$ as a *functor*. Let E, E' be two fibre spaces over X, f an X-homomorphism of E into E'. For any section s of E, the composed map fs is a section of E', continuous if s is continuous. *We get thus a map, noted* f, *of* $\mathrm{H}^0(X, E)$ *into* $\mathrm{H}^0(X, E')$. The usual functor properties are satisfied:

a. If the two fibre spaces are identical and f is the identity, the so is f.

b. If f is an X-homomorphism of E into E' and f' an X-homomorphism of E' into E'' (E, E', E'' fibre spaces over X) then $(f'f) = f'f$.

Let (X, E, p) be a fibre space, f a continuous map of a space X' into X, and E' the inverse image of E under f.

II. Sheaves of sets

Throughout this exposition, we will now use the word *"section"* for *"continuous section"*.

2.1 Sheaves of sets

Definition **2.1.1**. — *Let X be a space. A* sheaf of sets *on X (or simply a* sheaf*) is a fibre space (E, X, p) with base X, satisfying the condition: each point a of E has an open neighbourhood U such that p induces a homeomorphism of U onto an open subset $p(U)$ of X.*

This can be expressed by saying that p is an interior map and a local homeomorphism. It should be kept in mind that, even if X is separated, E is not supposed separated (and will in most important instances not be separated).

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2.2

2.3 Definition of a sheaf by systems of sets

2.4 Permanence properties

2.5 Subsheaf, quotient sheaf. Homeomorphism of sheaves

2.6 Some examples

a.

b.

c.

d. **Sheaf of germs of subsets**. Let X be a space, for any open set $U \subset X$ let $P(U)$ be the set of subsets of U. If $V \subset U$, consider the map $A \longrightarrow A \cap V$ of $P(U)$ into $P(V)$. Clearly the conditions of transitivity, and of proposition 2.3.1. corollary, are satisfied, so that the sets $P(U)$ appear as the sets $\mathrm{H}^0(U, P(X))$ of sections of a well determined sheaf on X, the elements of which are called *germs of sets in X*. Any condition of a local character on subsets of X defines a subsheaf of $P(X)$, for instance the sheaf of *germs* *of closed sets* (corresponding to the relatively closed sets in U), or if X is an analytic manifold, the sheaf of germs of analytic sets, etc.

Other important examples of sheaves will be considered in the next chapter.

III. Group bundles and sheaves of groups

IV. Fibre spaces with structure sheaf

V. The classification of fibre spaces with structure sheaf

SUR LES FAISCEAUX ALGÉBRIQUES ET LES FAISCEAUX ANALYTIQUES COHÉRENTS

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1 Généralités sur les faisceaux algébriques cohérents. (Rappels)

Soit X un espace topologique muni

ON COHERENT ALGEBRAIC AND ANALYTIC SHEAVES[1]

The aim of this talk is to generalise certain theorems of Serre. It makes fundamental use of the techniques of Serre [?, ?, ?].

1. Generalities on coherent algebraic sheaves

Let X be a topological space endowed with a sheaf of rings O. A sheaf of O-modules A (or simply an O-module) is said to be *of finite type* if, on every small-enough open subset, it is isomorphic to a quotient of O^n (for some finite integer $n \geq 0$), and *coherent* if it is of finite type and if, for every homomorphism $O^m \longrightarrow A$ on an open subset U of X, the kernel is of finite type. If $0 \longrightarrow A' \longrightarrow A \longrightarrow A'' \longrightarrow 0$ is an exact sequence of O-modules, and if two of the modules are coherent, then so too is the third; the kernel, cokernel, image, and coimage of a homomorphism of coherent O-modules is a coherent O-module. If A and B are coherent O-modules, then so too is the sheaf $_O(A,B)$ of germs of homomorphisms from A to B. If O itself is coherent, then coherent O-modules are exactly the O-modules that, on small-enough open subsets, are isomorphic to the cokernel of some homomorphism $O^m \longrightarrow O^n$. For all of this, and other elementary properties, see [?, chapitre 1, paragraphe 2].

Let X be an algebraic set (over an algebraically closed field k, to illustrate the idea; but the results of this talk still hold true for schemes, and even for general arithmetic schemes...). We denote by O_X the structure sheaf of X, with its sections over an open subset $U \subset X$

[1]Translated by T. Hosgood
https://thosgood.com/translations/

being the regular functions on U. This is a sheaf of rings, and even of k-algebras.

Theorem 1

(a) O_X is a coherent sheaf of rings.

(b) If X is affine with coordinate ring $A(X)$, then, for every coherent O-module A on X, the stalks A_x are generated by the canonical image of $\Gamma(X,A)$. Furthermore, $\Gamma(X,A)$ is an $A(X)$-module of finite type, and every $A(X)$-module of finite type comes from an essentially unique coherent O-module. (Recall that $\Gamma(X,A)$ denotes the module of sections of A over X).

(c) Under the conditions of b), we have that $\mathrm{H}^i(X,A) = 0$ for $i > 0$.

Proof. For the proofs, which are very elementary, see [?, chapitre 2, paragraphes 2,3,4], or an talk of Cartier in the 1957 *Séminaire Grothendieck*. □

0.1 A dévissage theorem

Definition 1 Let C be an abelian category, and C' a subclass of C. We say that C' is an *exact subcategory* if, for every exact sequence $0 \longrightarrow A' \longrightarrow A \longrightarrow A'' \longrightarrow 0$ in C with two (non-zero) terms in C', the third term is also in C', and if every direct factor of any $A \in C'$ is also in C'.

Theorem 2 Let X be an algebraic set; suppose that, for every irreducible subset Y of X, we are given a coherent O_Y-module F_Y on Y that has support equal to Y. Let $K(X)$ be the abelian category of coherent algebraic sheaves on X. Then every *exact* subcategory K of $K(X)$ containing the F_Y is identical to $K(X)$.

Proof. The proof is done by induction on $n = \dim X$, with the case $n = 0$ being immediate, by the second condition of Defintion 1. So suppose that $n > 0$, and that the theorem is true in dimension $< n$. We can consider $K(Y)$ as a subcategory of $K(X)$ (where Y is some given closed subset of X), and then $K \cap K(Y)$ is a subcategory of $K(Y)$ satisfying the conditions of Theorem 2, and so, if $\dim Y < n$, then the induction hypothesis implies that $K(Y) = K(Y) \cap K$, i.e. $K(Y) \subset K$.

Lemma 1 Let Y be a closed subset of X, and A a coherent O_X-module such that $\operatorname{supp} A \subset Y$. Let I_Y be the sheaf of ideals of O_X defined by Y. Then there exists an integer k such that $I_Y^k A = 0$.

Proof. By "compactness" reasons, we can restrict to the case where X is affine, and then apply part *b)* of Theorem 1, noting that, if A is defined by the $A(X)$-module $M = \Gamma(X, A)$, then the ideal of $\operatorname{supp} A$ is the intersection of the minimal prime ideals associated to the annihilator of M, whence the result. □

Corollary Under the above conditions, A admits a composition series with each composition factor A_i/A_{i+1} lying in $K(Y)$.

This implies that A_i/A_{i+1} is annihilated by I_Y; we take $A_i = I_Y^i A$. In the case where $\dim Y < n$, by induction on the length of this composition series, using Definition 1 and the fact that $K(Y) \subset K$, we see that, *if* $\dim \operatorname{supp} A < n$*, then* $A \in K$.

Suppose first of all that X is irreducible. For $A \in K(X)$, let $T(A)$ be the torsion submodule of A (whose stalks are the torsion submodules of A_x).

Lemma 2 If $A \in K(X)$, then the torsion submodule $T(A)$ is also in $K(X)$, and $A = T(A)$ if and only if $\operatorname{supp} A \neq X$.

Proof. We can immediately restrict to the case where X is affine, where it is evident, by the interpretation of coherent O-modules as $A(X)$-modules of finite type. □

Using the exact sequence $0 \longrightarrow T(A) \longrightarrow A \longrightarrow A_0 \longrightarrow 0$, and that $T(A) \in K$, we see that $A \in K$ if and only if $A_0 \in K$.

Let R be the sheaf of fields over X given by the fields of fractions of the $O_{X,x}$, i.e. the sheaf of germs of rational functions, which is a constant sheaf, and we have an injective homomorphism $A_0 \longrightarrow A_0 \otimes_{O_X} R$ Representing A_0 locally as the cokernel of a homomorphism $O_X^m \longrightarrow O_X^{m'}$, we see that the tensor product $A_0 \otimes_{O_X} R$ is locally isomorphic (as sheaves of R-modules) to a sheaf of the form R^k, and thus conclude that it is *globally* isomorphic to R^k thanks to:

Lemma 3 On any irreducible algebraic set X, every locally constant sheaf is constant.

Proof. This is an easy consequence of the fact that every open subset of X is connected (consider a maximal open subset where the sheaf in question is constant!). □

We will thus identify $A_0 \otimes_{O_X} R$ with some R^k, which contains the sub-O_X-module O_X^k. Consider the exact sequence

$$0 \longrightarrow A_0 \longrightarrow A_0 + O_X^k \longrightarrow Q \longrightarrow 0$$

where Q is defined as the cokernel of the injection homomorphism. We immediately see that $Q \otimes_O R = 0$, and so Q is a torsion module, and so $\operatorname{supp} Q \neq X$ (Lemma 2), whence $Q \in K$. (We implicitly make use of the fact that $A_0 + O_X^k$ is a coherent O-module, which can be easily verified). Then $A_0 \in K$ *if and only if* $A_0 + O_X^k \in K$. Similarly, the analogous exact sequence $0 \longrightarrow O_X^k \longrightarrow A_0 + O_X^k \longrightarrow Q' \longrightarrow 0$, where $\operatorname{supp} Q \neq X$, and whence $Q' \in K$, implies that $A_0 + O_X^k \in K$ *if and only if* $O_X^k \in K$. Finally, suppose that $k > 0$, i.e. that A is not a torsion O-module, i.e. that $\operatorname{supp} A = X$; then $O_X^k \in K$ *if and only if* $O_X \in K$, as follows immediately from Definition 1. So the above "if and only if"s imply that, if A is such that $\operatorname{supp} A \neq X$, then $A \in K$ *if and only if* $O_X \in K$. Taking $A = F_X$, we thus see that $O_X \in K$, whence every $A \in K(X)$ with support equal to X is in K, and since the same is true for the $A \in K(X)$ with suppose not equal to X, we indeed have that $K(X) \subseteq K$.

Now if X is not necessarily irreducible, let X_i be its irreducible components. For every coherent algebraic sheaf A on X, let A_i be the sheaf that agrees with A on X_i, and with 0 on $X \setminus X_i$; then A_i is a coherent O-module that can be identified with a quotient of A. We have a natural homomorphism $A \longrightarrow \coprod_i A_i$ from A to the direct sum of the A_i that is *injective*; let Q be its cokernel; we thus have an exact sequence $0 \longrightarrow A \longrightarrow \coprod_i A_i \longrightarrow Q \longrightarrow 0$. Since $\operatorname{supp} Q \subset \bigcup_{i \neq j} X_i \cap X_j$, we have that $Q \in K$; to prove that $A \in K$, it suffices to prove that $\coprod_i A_i \in K$, or even that each A_i is in K. But, by what we have already seen, applied to X_i, we have that $K(X_i) \in K$, whence we again conclude that $\operatorname{supp} A_i \subset X_i$ implies that $A_i \in K$, by using the Corollary of Lemma 1. This proves Theorem 2. □

Remark We say that the subcategory K of $K(X)$ is *left exact* if, for every exact sequence $0 \longrightarrow A' \longrightarrow A \longrightarrow A'' \longrightarrow 0$ in $K(X)$, we have that A' (respectively A) is in K provided that the two other terms are in K. The proof of Theorem 2 proves that the conclusion still holds true if we suppose that K is only left exact, *provided that* the F_Y considered as O_Y-modules are torsion free. This suffices to prove that, if X is complete, then the $\Gamma(X, A)$, for $A \in K(X)$, are vector spaces of finite dimension (since the category K of $A \in K(X)$ having this property is left exact, and contains the O_Y): this is the proof by Serre.

0.2 Complements on sheaf cohomology

Let X be a topological space, and write C^X to denote the category of abelian sheaves on X. We define, in the usual manner, injective sheaves, and we can prove the existence, for all $A \in C^X$, of a resolution $C(A)$ of A by injective sheaves, which allows us to develop the theory of right-derived functors. In particular, consider the left-exact functor $\Gamma(X,A)$ from C^X to the category C of abelian groups; its derived functors are denoted $\mathrm{H}^i(X,A)$. So

$$\mathrm{H}^i(X,A) = \mathrm{H}^i\big(\Gamma(X,C(A))\big).$$

The $\mathrm{H}^i(X,A)$ form a "cohomological functor" in A that is zero for $i < 0$, and satisfies

$$\mathrm{H}^0(X,A) = \Gamma(X,A).$$

If $f : X \longrightarrow Y$ is a continuous map from X to a space Y, then we can define, for any abelian sheaf B on Y, the abelian sheaf $f^{-1}(B)$ on X, which we call the *inverse image of* B, as well as the canonical homomorphism

$$\mathrm{H}^0(Y,B) \longrightarrow \mathrm{H}^0(X,f^{-1}(B))$$

which extends uniquely to give functorial, compatible (with the coboundary operators) homomorphisms

$$\mathrm{H}^i(Y,B) \longrightarrow \mathrm{H}^i(X,f^{-1}(B)).$$

Now let A be an abelian sheaf on X, and define its *direct image* $f_*(A)$ to be the abelian sheaf on Y whose sections over any open subset V are the sections of A over $f^{-1}(V)$. Clearly f_* is a covariant additive left-exact functor from C^X to C^Y, and, if Γ_X (resp. Γ_Y) denotes the "sections" functor on C^X (resp. C^Y), then, by definition

$$\Gamma_X = \Gamma_Y \circ f_*.$$

Furthermore, it is trivial to show that f_* sends injective sheaves to injective sheaves. From this, we easily obtain the *Leray spectral sequence of the continuous map* f, i.e. there is a cohomological spectral sequence starting with

$${}_2^{p,q} = \mathrm{H}^p(Y,\mathrm{R}^q f_*(A))$$

that abuts to $\mathrm{H}^\bullet(X,A)$, where the $\mathrm{R}^q f_*(A)$ are the sheaves on Y given by taking the right-derived functors of the functor $f_*\colon C^X \longrightarrow C^Y$, i.e. $\mathrm{R}^q f_*(A) = \mathrm{H}^q(f_* C(A))$. We immediately see that $\mathrm{R}^q f_*(A)$ is *the sheaf on Y associated to the presheaf* $V \mapsto \mathrm{H}^q(f^{-1}(V),A)$.

From the Leray spectral sequence, we get homomorphisms

$$\mathrm{H}^p(Y, f_*(A)) \longrightarrow \mathrm{H}^p(X,A) \tag{1}$$

whose direct definition is evident (noting that we have a natural homomorphism $f^{-1}(f_*(A)) \longrightarrow A$). Furthermore, *if* $\mathrm{R}^q f_*(A) = 0$ *for* $q > 0$, *then the homomorphisms in (1) are isomorphisms*. This follows immediately from the spectral sequence, or, even more simply, from the fact that $f_*(C(A))$ is an injective resolution of $f_*(A)$.

For the results of this section, see the 1957 *Séminaire Grothendieck*.

0.3 Supplementary results on algebraic sheaves on projective space

Let V be a finite-dimensional k-vector space, and $\mathbf{P}$ the associated projective space, the quotient of $V \setminus \{0\}$ by the algebraic group $k^\times = k \setminus \{0\}$. We see that $V \setminus \{0\}$ is a principal algebraic $k^\times$-bundle on $\mathbf{P}$, and so it defines an associated vector bundle on $\mathbf{P}$, with fibres of dimension 1; the sheaf of germs of regular sections of the *dual* bundle is denoted $O(1)$, and we denote by $O(n)$ the n-fold tensor product of $O(1)$ with itself if $n \geq 0$, and the $(-n)$-fold tensor product of the dual sheaf if $n < 0$ (in particular then, $O(0) = O_{\mathbf{P}}$). If A is an algebraic sheaf on $\mathbf{P}$, then we let $A(n) = A \otimes_{O_{\mathbf{P}}} O_n$, and so $A(m)(n) = A(m+n)$. The definitions of $O(n)$ and of the operation $A \mapsto A(n)$ can immediately be extended to sheaves on a product $\mathbf{P} \times Y$, where Y is an arbitrary algebraic set.

Theorem 3

(a) Let Y be an affine algebraic set, and A a coherent algebraic sheaf on $\mathbf{P} \times Y$. Then, for every n large enough, $A(n)$ is generated by the module of its sections, i.e. $A(n)$ is isomorphic to some quotient of $O^k_{\mathbf{P}\times Y}$, for some integer k.

(b) For n large enough, $\mathrm{H}^i(\mathbf{P}, O(n)) = 0$.

Proof. The proof is elementary; for a), see [?, page 247, théorème 1] (where the proof is given for the case where Y is a single point, but the same method works for arbitrary Y), and for b) see [?, page 259, théorème 2]. We could also give a direct proof of b) by calculating $\mathrm{H}^i(\mathbf{P}, O(n))$ using the Čech method, which can be applied here, by part (c) of Theorem 1 (see the *Séminaire Grothendieck* for more on this point), and using the well-known cover of $\mathbf{P}$ by $(r+1)$ affine open subsets, each isomorphic to k^r. □

Now suppose that $k = \mathbf{C}$ is the field of complex numbers, so that $\mathbf{P}$ is also endowed with the structure of an analytic space, which we denote by $\mathbf{P}^h$; this is itself endowed with a sheaf of analytic local rings, which we denote by O^h; finally, we can define, as above, the sheaves $O^h(n)$. With this, we have:

Corollary

(a) Let A^h be a coherent O^h-module on $\mathbf{P}^h$. Then, for all n large enough, $A^h(n)$ is isomorphic to a quotient of $(O^h)^k$, for some integer k.

(b) For n large enough, $\mathrm{H}^i(\mathbf{P}^h, O^h(n)) = 0$.

Proof. The proof is distinctly deeper: see [?, lemme 5, page 12, and lemma 8, page 24]. It works by induction on the dimension, and makes essential use of the fact that the cohomology $\mathbf{P}^h$ with values in a coherent O^h-module is of finite dimension. □

0.4 The finiteness theorem: statement

Let $f: X \longrightarrow Y$ be a regular map of algebraic sets, and let A be an algebraic sheaf, i.e. an O_X-module, on X. Then its direct image by f, and, more generally, by the $\mathrm{R}^q f_*(A)$ (see §3), are O_Y-modules. In the case where A is coherent (or, more generally, "quasi-coherent", in the sense of Cartier, *Séminaire Grothendieck*), we can easily show that, for every *affine* open subset V of Y,

$$\Gamma(V, \mathrm{R}^q f_*(A)) = \mathrm{H}^q(f^{-1}(V), A)$$

and that the sheaves $\mathrm{R}^q f_*(A)$ are "quasi-coherent". We will given sufficient conditions for them to be coherent.

Definition 2 A morphism $f: X \longrightarrow Y$ of algebraic sets is said to be *proper* if, for every irreducible component X_i of X, the scheme X_i is complete over the scheme $f(X_i)$ (see the 1955/56 *Séminaire Cartan-Chevalley*).

A more geometric definition is the following: f is proper if, for every algebraic set Z, the corresponding map $X \times Z \longrightarrow Y \times Z$ is *closed*. Let $f: X \longrightarrow Y$ and $g: Y \longrightarrow Z$ be morphisms of algebraic sets; if f and g are proper, then gf is proper; if gf is proper, then f is proper, and g is proper if further the image of f is dense in Y. For X to be complete, it is necessary and sufficient for the morphism from X to an algebraic set consisting of a single point to be proper. If X is a locally closed subset of a complete variety X', then for $f: X \longrightarrow Y$ to be proper, it is necessary and sufficient for its graph to be closed. Combining this with Chow's lemma (§7, Lemma 4), the fact that an algebraic subset of an algebraic set over the complex numbers is closed if and only it is closed for the topology of the underlying space [?, proposition 7, page 12], and the fact that a complex projective space is compact, we easily conclude, from the above criterion that, in the "classical case", a morphism is proper if and only if the map of underlying analytic spaces is proper in the usual sense (i.e. the inverse image of a compact subset being compact); compare with [?, proposition 12, proposition 6], where a particular case is proven: X is complete if and only if it is compact.

Theorem 4 Let $f: X \longrightarrow Y$ be a proper morphism of algebraic sets. For any coherent algebraic sheaf A on X, the algebraic sheaves $\mathrm{R}^q f_*(A)$ on Y (and, in particular, the direct image $f_*(A)$) are coherent.

Proof. The proof will be given in §7. □

We state here the following corollary, obtained by taking Y to be a single point:

Corollary Let A be a coherent algebraic sheaf on a complete algebraic set. Then the $\mathrm{H}^i(X, A)$ are vector spaces of finite dimension.

0.5 An algebraic-analytic comparison theorem: statement

Let X be an algebraic set over the field of complex numbers, and denote by X^h the underlying analytic set (see [?] for proper definitions), and by O^h or O^h_X the sheaf of (analytic) local rings of X^h. The identity map $i_X : X^h \longrightarrow X$ is continuous, and we can thus consider the inverse image $i^{-1}(O_X)$, and we have a natural homomorphism of sheaves of rings $i^{-1}(O_X) \longrightarrow O_X^h$, which allows us to consider O_X^h as a sheaf of algebras over $i^{-1}(O_X)$. If now A is an O_X-module, then $i^{-1}(A)$ is an $i^{-1}(O_X)$-module, and we set

$$A^h = i^{-1}(A) \otimes_{i^{-1}(O_X)} O_X^h$$

where A^h is called the *analytic sheaf associated to A*. It is shown in [?] that the covariant functor $A \longrightarrow A^h$ is *exact*. We have a functorial homomorphism

$$i^{-1}(A) \longrightarrow A^h$$

which is injective, and gives homomorphisms (see §3)

$$\mathrm{H}^i(X, A) \longrightarrow \mathrm{H}^i(X^h, A^h). \tag{2}$$

We will see that, if X is complete, then the homomorphisms in (2) are isomorphisms. However, we will actually prove a more general result. Let

$$f : X \longrightarrow Y$$

a morphism of algebraic sets; consider the map $f^h : X^h \longrightarrow Y^h$. From the commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle i_X}\downarrow & & \downarrow{\scriptstyle i_Y} \\ X^h & \xrightarrow{f^h} & Y^h \end{array}$$

we easily obtain a functorial homomorphism

$$i_Y^{-1}(f_*(A)) \longrightarrow f_*(i_X^{-1}(A))$$

for any sheaf A on X; if A is an O_X-module, then the canonical homomorphism $i_X^{-1}(A) \longrightarrow A^h$ also defines a homomorphism

$$f_*^h(i_X^{-1}(A)) \longrightarrow f_*^h(A^h).$$

The composition $i_Y^{-1}(f_*(A)) \longrightarrow f_*^h(A^h)$ of these homomorphisms is compatible with the canonical homomorphism $i_Y^{-1}(O_Y) \longrightarrow O_Y^h$ of rings of operators, whence, by tensoring with a canonical homomorphism, we obtain

$$f_*(A)^h \longrightarrow f_*^h(A^h). \tag{3}$$

This functorial homomorphism can be extended, in a unique way, to functorial homomorphisms (that commute with the coboundary operators):

$$(\mathrm{R}^q f_*(A))^h \longrightarrow \mathrm{R}^q f_*^h(A^h). \tag{4}$$

These homomorphisms have all the functorial properties that we might desire, but whose precise statements will not be given here (even though they will, of course, be essential in the proofs.)

Theorem 5 Suppose that the morphism of algebraic sets $f: X \longrightarrow Y$ is proper. Then the homomorphisms in (4) are isomorphisms.

Proof. The proof will be given in the following section. □

Taking Y to be a single point, we obtain the following:

Corollary 1 If X is a complete algebraic set, then the homomorphisms in (2) are isomorphisms.

Since $A \longrightarrow A^h$ sends coherent algebraic sheaves to coherent analytic sheaves (an immediate consequence of the exactness of the functor), the combination of Theorem 4 and Theorem 5 gives:

Corollary 2 Under the conditions of Theorem 5, the $f^h(A^h)$ are coherent analytic sheaves.

It is very plausible that, more generally, if $g: V \longrightarrow W$ is a proper holomorphic map of analytic spaces, and if F is a coherent analytic sheaf on V, then $g_*(F)$ is a coherent analytic sheaf. This is indeed true if the sets $f^{-1}(y)$ (for $y \in W$) are finite (as we can see by a classical theorem of Oka; see the 1953/54 *Séminaire Cartan*), or if W consists of a single point (by a result of Serre-Cartan, *loc. cit.*).

Corollary 3 Under the conditions of Theorem 5, suppose further that Y is an *affine* algebraic set, and let $A(Y)$ (resp. $A^h(Y)$) be the ring of regular functions on Y (resp. the ring of holomorphic functions on Y^h). Then there is a canonical isomorphism

$$\mathrm{H}^q(X^h, A^h) = \mathrm{H}^q(X, A) \otimes_{A(Y)} A^h(Y). \tag{5}$$

Proof. We have already said that $\mathrm{H}^q(X,A)$ can be identified with the module of sections of $\mathrm{R}^q f_*(A)$ over Y, and, similarly, we say that $\mathrm{H}^q(X^h,A^h)$ can be identified with the module of sections of $\mathrm{R}^q f_*^h(A^h)$ over Y. To prove this, it suffices to use the Leray spectral sequence of f^h (see §3), and to note that, identifying Y^h with a closed analytic subset of some $\mathbf{C}^n$, its cohomology with values in the coherent analytic sheaves $\mathrm{R}^q f_*^h(A^h)$ is zero in dimensions > 0, by a fundamental theorem of Cartan (see the 1951/52 *Séminaire Cartan*). It thus suffices, by Theorem 5, to prove that, if B is a coherent O_Y-module on an affine variety Y, then

$$\mathrm{H}^0(Y^h,B) = \mathrm{H}^0(Y,B) \otimes_{A(Y)} A^h(Y). \tag{6}$$

But we note that both sides of this equation are exact functors in B, which means we only need to verify (6) in the case where $B = O_Y$, but then it is trivial. □

0.6 Proof of Theorems 4 and 5

The proofs follow mainly from Theorem 3, the "dévissage" of Theorem 2 (which is necessary since there is no reason for X to be isomorphic to a locally closed subset of a projective space), and the following:

Lemma 4 *(Chow's lemma.)* — Let X be a an irreducible algebraic set Then there exists an algebraic set X' that is locally closed in some projective space $\mathbf{P}$, and a proper birational morphism $g: X' \longrightarrow X$.

Recall (§5) that "proper" implies, in this case, that the graph of g is a *closed* subset of $\mathbf{P} \times X$. Here we only make use of the fact that g is *proper* and *surjective*.

Proof. We cover X by affine open subsets X_i, with each X_i locally closed in some projective space $\mathbf{P}_i$, whence we have a diagonal map $\bigcap X_i \longrightarrow \coprod \mathbf{P}_i$. We take X' to be the closure in $X \times \coprod \mathbf{P}_i$ of the graph of this diagonal map (or, really, X' is its normalisation).

Theorem 4 and Theorem 5 say that every coherent algebraic sheaf A on X satisfies a certain property. But we immediately see that, in both cases, the class K of the $A \in K(X)$ having the property in question is an *exact* subcategory (§2, Definition 1), by using the exact sequence of the $\mathrm{R}^q f_*$ (and the $\mathrm{R}^q f_*^h$) corresponding to an exact sequence of sheaves $0 \longrightarrow A' \longrightarrow A \longrightarrow A'' \longrightarrow 0$, and by using either the fact that, if in an exact sequence of

O_X-modules $A \longrightarrow B \longrightarrow C \longrightarrow D \longrightarrow E$, the four outer terms are coherent, then so too is C, or (in the case of Theorem 5) the classical 5 lemma. By Theorem 2, it thus suffices to find, for every irreducible closed subset Z of X, a coherent algebraic sheaf *on* Z, with support equal to Z, and belonging to K. Note that the restriction of f to Z is again proper, and so we can assume that $Z = X$, which means it suffices to find *one* coherent O_Y-module A, with support equal to X, such that $A \in K$. Consider the morphism $f' \colon X' \longrightarrow X$ described in Lemma 4. Since X' is embedded into $\mathbf{P}$, we can consider the sheaves $O_{X'}(n)$ on X' given by reducing the $O_{\mathbf{P}}(n)$ (see §4) modulo the sheaf of ideals defined by X' in $\mathbf{P}$. We claim that, for n large enough, the sheaf $A = f(O_{X'}(n))$ is in K (which will finish the proof, since the support of this sheaf is clearly equal to X). This will follow from:

Lemma 5 Let $g \colon V \longrightarrow W$ be a *proper* morphism of algebraic sets, with V a locally closed subset of a projective space $\mathbf{P}$. Let G be a coherent algebraic sheaf on V. Then, for n large enough,

$$\mathrm{R}^p f_*(G(n)) = 0$$

for $p > 0$, and $f_*(G(n))$ is coherent. Furthermore, if $k = \mathbf{C}$, then

$$\mathrm{R}^p f_*^h(G(n)^h) = 0$$

for $p > 0$, and

$$(f_*(G(n)))^h \longrightarrow f_*^h(G(n)^h)$$

is an isomorphism.

First we will show how this lemma will imply the previous one. Applying the lemma to $f' \colon X' \longrightarrow X$, we immediately see, from the definitions, and from the fact that $\mathrm{R}^p f_*(O(n)) = 0$ for $p > 0$, that

$$\mathrm{R}^p(ff')_*(O(n)) = \mathrm{R}^p f_*(f'(O(n))) = \mathrm{R}^p f_*(A).$$

But the first object is zero for $p > 0$ and large enough n, by Lemma 5 applied to $ff' \colon X' \longrightarrow Y$, and so $\mathrm{R}^p f_*(A) = 0$ for $p > 0$, and, a fortiori, $\mathrm{R}^p f_*(A)$ is coherent for $p > 0$; similarly, $f_*(A)$ is coherent, since $f_*(A) = (ff')_*(O(n))$, and so it suffices to apply Lemma 5 to ff'. This thus proves that $A \in K$ in the setting of Theorem 4. In the setting of Theorem 5, the same argument proves that, if n is large enough, $\mathrm{R}^p f_*^h(A^h) = 0$ for $p > 0$, and, a fortiori, the homomorphisms in (4) are isomorphisms for $q > 0$; similarly, the homomorphism

$(f_*(A))^h \longrightarrow f_*^h(A^h)$ is an isomorphism, since both the domain and codomain can be identified (respectively) with $((ff')_*(O(n)))^h$ and $f^h(A^h) = f_*'^h(O(n)^h)$ (since $A^h = (f'(O(n)))^h = f_*'^h(O(n)^h)$, by Lemma 5 applied to $ff' \colon X' \longrightarrow Y$).

It thus remains only to prove Lemma 5. Since the graph V' of g is a closed subset of $\mathbf{P} \times W$, isomorphic to V, we can, by identifying sheaves on V with sheaves on V' (and thus on $\mathbf{P} \times W$), suppose that $V = \mathbf{P} \times W$, and that g is the projection homomorphism. Furthermore, we can suppose that W is affine, and even that $W = k^m$.

We first prove Lemma 5 in the case where $F = O_k$. For an arbitrary field k, this thus implies that $\mathrm{H}^p(\mathbf{P} \times W, O(n)) = 0$ for $p > 0$ and n large enough, and that $\mathrm{H}^0(\mathbf{P} \times W, O(n))$ is a module of finite type over the coordinate ring $A(W)$ of W. Since $O_{\mathbf{P}\times W}(n)$ is the "tensor product" (in the sense of algebraic sheaves) of the sheaves $O_{\mathbf{P}}(n)$ on $\mathbf{P}$ and O_W on W, the Künneth formula (whose proof, in this setting, is elementary) applies, and we thus obtain the stated result, taking into account the fact that $\mathrm{H}^i(W, O) = 0$ for $i > 0$ (part (c) of Theorem 1) and part (b) of Theorem 3, since then

$$\mathrm{H}^i(\mathbf{P} \times W, O_{\mathbf{P}\times W}(n)) = \mathrm{H}^i(\mathbf{P}, O_{\mathbf{P}}(n)) \otimes_F A(W)$$

is zero for $i > 0$ and n large enough, and is of finite type over $A(W)$ when $i = 0$, since $\mathrm{H}^0(\mathbf{P}, O_{\mathbf{P}}(n))$ is clearly of finite dimension. When $k = \mathbf{C}$, we must prove that, for n large enough,

$$\mathrm{H}^i(\mathbf{P}^h \times W', O(n)^h) = 0$$

for $i > 0$ and W' any Stein open subset of W^h, and also that $f_*(O(n)^h)$ can be identified with $(f_*(O(n)))^h$, i.e. with $\mathrm{H}^0(\mathbf{P}, O(n)) \otimes O_W^h$; or, in other words, that

$$\mathrm{H}^i(\mathbf{P}^h \times W', O(n)) = \mathrm{H}^0(\mathbf{P}, O(n)) \otimes \mathrm{H}^0(X', O_W^h)$$

for every Stein open subset W' of W. But $H^\bullet(\mathbf{P}^h \times W', O(n)^h)$ can be calculated by a *vectorial-topological variant of the Künneth theorem* (using the fact that the space $\mathrm{H}^0(W', O_{W'}^h)$ is *nuclear*; see the 1953/54 *Séminaire Schwartz*); taking into account the fact that $\mathrm{H}^i(W', O_W) = 0$ for $i > 0$, we see that it is equal to $\mathrm{H}^i(\mathbf{P}^h, O(n)^h) \otimes \mathrm{H}^0(W', O_w)$, by a fundamental theorem of Cartan concerning Stein varieties (which, for our purposes here, it suffices to know for a polycylinder and the structure sheaf. where it is an easy consequence of the aforementioned vectorial-topological Künneth theorem). The above claims then follow from corollary (b) of Theorem 3, taking into account the fact that $\mathrm{H}^0(\mathbf{P}^h, O(n)^h) = \mathrm{H}^0(\mathbf{P}, O(n))$ (which is proven in the proof of that corollary).

To prove Lemma 5 in the general case, we proceed by induction on p, since the lemma is trivial for p large enough, for dimension reasons. By part (a) of Theorem 3, A is isomorphic to a quotient of some $O(m)^k = L$, i.e. we have an exact sequence $0 \longrightarrow A' \longrightarrow L \longrightarrow A \longrightarrow 0$, whence, for all n, an exact sequence

$$0 \longrightarrow A'(n) \longrightarrow L(n) \longrightarrow A(n) \longrightarrow 0$$

which gives an exact sequence

$$\mathrm{R}^p f_*(A'(n)) \longrightarrow \mathrm{R}^p f_*(L(n)) \longrightarrow \mathrm{R}^p f_*(A(n)) \longrightarrow \mathrm{R}^{p+1} f_*(A'(n)).$$

By the induction hypothesis, the last term in this sequence is zero for n large enough, and so too is $\mathrm{R}^p f_*(L(n))$ when $p > 0$, by what we have already proven, whence $\mathrm{R}^p f_*(A(n)) = 0$ for n large enough and $p > 0$. If $p = 0$, then the same exact sequence proves that, for n large enough, $f_*(A(n))$is coherent, since $f_*(L(n))$ is coherent, and $f_*(A'(n))$ is anyway quasi-coherent. In the case where $k = \mathbf{C}$, we can prove, in the same way, that $\mathrm{R}^p f_*^h(A(n)^h) = 0$ for n large enough and $p > 0$. It remains only to show that, for n large enough, $(f_*(A(n)))^h \longrightarrow f^h(A^h)$ is bijective. For this, we write A as the cokernel of a homomorphism $L' \longrightarrow L$, where L and L' are isomorphic to direct sums of sheaves of the form $O(m)$ for various m (which is possible by part (a) of Theorem 3). By the above, for n large enough $f_*(A(n))$ and $f_*^h(A(n)^h)$ can be identified (respectively) with the cokernel of $f_*(L'(n)) \longrightarrow f(L(n))$ and the cokernel of $f^h(L'(n)^h) \longrightarrow f^h(L(n)^h)$; taking into account the fact that the functor $B \longrightarrow B^h$ is exact, we thus obtain a homomorphism of exact sequences

$$\begin{array}{ccccccc}
(f_*(L'(n)))^h & \longrightarrow & (f_*(L(n)))^h & \longrightarrow & (f_*(A(n)))^h & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \\
f_*^h * (L'(n)^h) & \longrightarrow & f_*^h * (L(n)^h) & \longrightarrow & f_*^h * (A(n)^h) & \longrightarrow & 0
\end{array}$$

Since, for n large enough, the first two vertical arrows are isomorphisms, so too is the third, by the five lemma, which finishes the proof. □

Remark The last paragraph of this proof can be simplified if we use the fact that A admits a finite resolution by sheaves that are direct sums of sheaves of the form $O(m)$ for various m;

but this fact is less elementary than part (a) of Theorem 3, and so we wanted to avoid using it.

0.7 Algebraic and analytic sheaves on a compact algebraic variety

We are going to complete Corollary 1 of Theorem 5:

Theorem 6 Let X be a complete algebraic set over **C**. Then every coherent analytic sheaf F on X^h is isomorphic to a sheaf A^h, where A is an essentially unique coherent algebraic sheaf on X.

The uniqueness of A follows from:

Corollary 1 With X as above, let A and B be coherent algebraic sheaves on X. Then the natural homomorphism

$$ {}_{O_X}(A,B) \longrightarrow {}_{O_X^h}(A^h,B^h) \tag{7} $$

is bijective.

Proof. This homomorphism comes from, by taking sections, the monomorphism of sheaves

$$ i_X^{-1}({}_{O_X}(A,B)) \longrightarrow {}_{O_X^h}(A,B) $$

(where denotes the sheaf of germs of homomorphisms), but we already know that

$$ ({}_{O_X}(A,B))^h = {}_{O_X^h}(A^h,B^h) \tag{8} $$

(an almost immediate consequence of the fact that $C \longrightarrow C^h$ is exact), and so, by applying Corollary 1 of Theorem 5 to the sheaf ${}_{O_X}(A,B)$ with $i = 0$, the result follows. □

From Corollary 1 and the exactness of the functor $C \longrightarrow C^h$ also follows the fact that, if F and G are coherent analytic sheaves on X that come from algebraic sheaves, and if u is a homomorphism from F to G, then the kernel, cokernel, image, and coimage of u all also come from algebraic sheaves. In particular, if X is embedded into a projective space **P**, then every coherent analytic sheaf on X is isomorphic to the cokernel of a homomorphism

$L^h \longrightarrow L'^h$, where L and L' are direct sums of finitely many sheaves of the form $O(k)$ (part (a) of the Corollary of Theorem 3); it thus follows that Theorem 6 is also true if X is *projective* (Serre]Theorem 6.

Let $0 \longrightarrow F' \longrightarrow F \longrightarrow F'' \longrightarrow 0$ be an exact sequence of coherent analytic sheaves on X^h, and suppose that F' and F'' come from coherent algebraic sheaves; we then claim that so too does F. Suppose that $F' = A'^h$ and $F'' = A''$, where A' and A'' are coherent algebraic sheaves; it suffices to show that the set $\mathrm{Ext}^1_{O_X}(X; A'', A')$ of classes of O-module extensions of A'' by A' can be identified with the analogous set $\mathrm{Ext}^1_{O_X^h}(X^h; A''^h, A'^h)$. But, more generally, we have canonical homomorphisms

$$\mathrm{Ext}^i_{O_X}(X; A'', A') \longrightarrow \mathrm{Ext}^i_{O_X^h}(X; A''^h, A'^h) \tag{9}$$

(defined without any restrictions on X, A', or A''), which are here isomorphisms, as follows from the spectral sequence of Ext of sheaves of modules (see the 1957 *Séminaire Grothendieck*), from the elementary local relations

$$({}^i_{O_X}(A,B))^h =^i_{O_X^h} (A^h, B^h) \tag{10}$$

that generalise (8) (with denoting the *sheaf* Exts), and from Corollary 1 of Theorem 5; this implies that the initial pages of the spectral sequences of both the domain and codomain of the morphism in (9) are identical.

Proof of Theorem 6. We can now prove Theorem 6, by induction on $n = \dim X$, with the theore]Theorem 6 being trivial when $n = 0$. So suppose that $n > 0$, and that the theorem is true in dimensions $< n$. Proceeding as in the end of the proof of Theorem 2, we can restrict to the case where X is irreducible. So consider the map $f : X' \longrightarrow X$ considered in Chow's lemma (Lemma 4), with X' a *projective* variety, and f a *birational* morphism. For every analytic sheaf F on X, let

$$F' = f^{-1}(F) \otimes_{f^{-1}(O_X^h)} O_X^h$$

(where the tensor product makes sense, since O_X^h is a module over $f^{-1}(O_X^h)$, which can be identified with a subsheaf (of rings) of O_X^h). It is easy to prove that, if F is coherent, then so too is F'. Furthermore, there is a natural homomorphism

$$F \longrightarrow f_*^h(F')$$

and, in the current setting, this homomorphism is bijective outside of an algebraic set Y of dimension $< n$ (where Y is the set of points of X over which f is not biregular). We thus have an exact sequence

$$0 \longrightarrow T \longrightarrow F \longrightarrow f_*^h(F') \longrightarrow T' \longrightarrow 0$$

where T and T' have support contained inside Y. Using the analogue of Lemma 1 of §2 (thanks to the compactness of X), we find that T (and even T') admits a composition series with composition factors that are coherent analytic sheaves *on* Y. These quotients are in fact "algebraic", by the induction hypothesis; thus so too are their extensions T and T'. Furthermore, since X' is projective, F' is also "algebraic", by what we have already said, and thus so too is $f_*^h(F')$, by Theorem 5 applied to $f : X' \longrightarrow X$ and $F' = B^h$. Thus the kernel of $f_*^h(F') \longrightarrow T'$ is also algebraic, and thus so too is F, which is an extension of this kernel by T. Thus we have proved Theorem 6. $\square$

Sur une note de Mattuck-Tate

1. Dans un travail récent [**?**], Mattuck et Tate déduisent l'inégalité fondamentale de A. Weil qui établit l'hypothèse de Riemann pour les corps de fonctions [**?**] comme conséquence facile du théorème de Riemann-Roch pour les surfaces. En essayant de comprendre la portée exacte du leur méthode, je suis tombé sur l'énoncé suivant, connu en fait depuis 1937 [**?**] [**?**] [**?**] (comme me l'a signalé J. P. Serre), mais apparemment peu connu et utilisé:

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2. Nous allons déduire sur X, nous désignerons par $l(D)$ la dimension de l'espace vectoriel des fonctions f sur X telles que $(f) \geq -D$ donc $l(D)$ ne dépend que de la classe de D. Rappelons *l'inégalité de Riemann-Roch*

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3. Ce qui précède n'utilisait pas à proprement parler la méthode de Mattuck-Tate (si ce n'est en utilisant l'inegalité de Riemann-Roch sur les surfaces). Nous allons indiquer maintenant comment la méthode de ces auteurs, convenablement généralisée, donne d'autres inégalités que celle de A. Weil. Nous nous appuierons sur le

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Remarques. Le corollaire 1 devient faux si on ne fait pas l'hypothèse que $K/2$ est encore une classe de diviseurs. En effet, toutes les hypothèses sauf cette dernière sont vérifiées si X est un surface non singulière *rationnelle*. Or, à partir d'une telle surface, on construit facilement une surface birationnellement équivalente par éclatements successifs, dont l'index τ soit < 0 (contrairement à (3.7 ter)). En effet, on vérifie aisément que lorsqu'on fait éclater un point dans une surface non singulière projective, l'index diminue d'une unité. (Cette remarque, ainsi que l'interprétation de l'inégalité (3.7) a l'aide de l'index, m'a été signalée par J. P Serre).

La disparité des énoncés qu'on déduit du théorème (3.2) est due au fait qu'il n'est pas relatif à un élément arbitraire de l'espace vectoriel E de Néron-Séveri introduit plus haut, mais à un élément du "lattice" provenant des diviseurs sur X. On notera d'ailleurs que dans le cas particulier où X est le produit des deux courbes C et C', le théorème 3.2 ne contient rien de plus que l'inégalité de A. Weil.

The cohomology theory of abstract algebraic varieties

It is less than four years since cohomological methods (i.e. methods of Homological Algebra) were introduced into Algebraic geometry in Serre's fundamental paper [?], and it seems already certain that they are to overflow this part of mathematics in the coming years, from the foundations up to the most advanced parts. All we can do here is to sketch briefly some of the ideas and results. None of these have been published in their final form, but most of them originated in or were suggested by Serre's paper.

Let us first give an outline of the main topics of cohomological investigation in Algebraic geometry, as they appear at present. The need of a theory of cohomology for 'abstract' algebraic varieties was first emphasized by Weil, in order to be able to give a precise meaning to his celebrated conjectures in Diophantine geometry [?]. Therefore the initial aim was to find the *'Weil cohomology' of an algebraic variety*, which should have as coefficients something 'at least as good' as a field of *characteristic 0*, and have such formal properties (e.g. duality, Künneth formula) as to yield the analogue of Lefschetz's 'fixed-point formula'. Serre's general idea has been that the usual 'Zariski topology' of a variety (in which the closed sets are the algebraic subset) is a suitable one for applying methods of Algebraic Topology. His first approach was hoped to yield at least the right Betti numbers of a variety, it being evident from the start that it could not be considered as the Weil cohomology itself, as the coefficient field for cohomology was the ground field of a variety, and therefore not in general of characteristic 0. In fact, even the hope of getting the 'true' *Betti numbers* has failed, and so have other attempts of Serre's [?] to get Weil's cohomology by taking the cohomology of the variety with values, not in the sheaf of local rings themselves, but in the sheaves of Witt-vectors constructed on the latter. He gets in this way modules over the ring $W(k)$ of infinite Witt vectors on the ground field k, and $W(k)$ is a ring of characteristic 0 even if k is of characteristic $p \neq 0$. Unfortunately, modules thus obtained over $W(k)$ may be infinitely generated, even when the variety V is an abelian variety [?]. Although interesting relations must certainly exist between these cohomology groups and the 'true ones', it seems certain now that the Weil cohomology has to be defined by a completely different approach. Such an approach was recently suggested to me by the *connections between sheaf-theoretic cohomology and cohomology of Galois groups on the one hand, and the classification of unramified coverings of a variety on the other* (as explained quite unsystematically in Serre's tentative Mexico paper [?]), and by Serre's idea that a 'reasonable' algebraic principal fiber space with structure group G, defined on a variety V, if it is not locally trivial, should become locally trivial on some covering of V *unramified* over a given point of V. This has been the starting point of a definition of the Weil cohomology (involving both 'spatial' and Galois cohomology), which seems to be the right one, and which gives clear suggestions how Weil's conjectures may be attacked by the machinery of Homological algebra. As I have not begun theses investigations seriously as yet,and as moreover this theory has a quite distinct flavor from the one of the theory of algebraic coherent sheaves which we shall now be concerned with, we shall not dwell any longer on Weil's cohomology. Let us merely remark that the definition alluded to has already been the starting-point of a theory of cohomological dimension of fields, developed recently by Tate [?].

The second main topic for cohomological methods is the *cohomology theory of algebraic coherent sheaves*, as initiated by Serre. Although inadequate for Weil's purposes, it is at present yielding a wealth of new methods and new notions, and gives the key even for results which were not commonly thought to be concerned with sheaves, still less with cohomology, such as Zariski's theorem on 'holomorphic functions' and his 'main theorem' - which can be stated now in a more satisfactory way, as we shall see, and proved by the same uniform elementary methods. The main parts of the theory, at present, can be listed as follows:

(a) General finiteness and asymptotic behaviour theorems.

(b) Duality theorems, including (respectively identical with) a cohomological theory of residues.

(c) Riemann-Roch theorem, including the theory of Chern classes for algebraic coherent sheaves.

(d) Some special results, concerning mainly abelian varieties.

The third main topic consists in the *application of the cohomological methods to local algebra*. Initiated by Koszul and Cartan-Eilenberg in connection with Hilbert's 'theorem of syzygies', the systematic use of these methods is mainly due again to Serre. The results are the *characterization* of regular local rings as those whose global cohomological dimension is finite, the clarification of *Cohen-Macaulay's equidimensionality theorem* by means of the notion of *cohomological codimension* [**?**], and specially the possibility of giving (for the first time as it seems) a *theory of intersections*, really satisfactory by its algebraic simplicity and its generality. Serre's result just quoted, that regular local rings are the the only ones of finite global cohomological dimension, accounts for the fact that only for such local rings does a satisfactory theory of intersections exist. I cannot give any details here on these subjects, nor on various results I have obtained by means of a *local duality theory*, which seems to be the tool which is to replace differential forms in the case of unequal characteristics, and gives, in the general context of commutative algebra, a clarification of the notion of residue, which as yet was not at all well understood. The motivation of this latter work has been the attempt to get a global theory of duality in cohomology for algebraic varieties admitting arbitrary singularities, in order to be able to develop intersection formulae for cycles with arbitrary singularities, in a non-singular algebraic variety, formulas which contain also a 'Lefschetz formula mod p' [**?**]. In fact, once a proper local formalism is obtained, the global statements become almost trivial. As a general fact, it appears that, to a great extent, the 'local' results already contain a global one; more precisely, global results on varieties of dimension n can frequently be deduced from corresponding local ones for rings of Krull dimension $n + 1$.

We will therefore

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Tapis de Quillen

Relation entre catégories et ensembles semi-simpliciauxA toute catégorie C, on associe un ensemble semi-simplicial $S(C)$, trouvant ainsi un foncteur pleinement fidèle

$$S : \longrightarrow .$$

Les systèmes locaux d'ensemble sur SC correspondent aux foncteurs sur C qui transforment toute flèche en isomorphisme (i.e. qui se factorisent par le groupoïde associé à C). Les H^i sur SC d'un tel système local (H^0 pour ensembles, H^1 pour groupes, H^i quelconques pour groupes abéliens) s'interprètent en termes des foncteurs $\varprojlim^{(i)}$ dérivés de $\varprojlim$, ou si on préfère, des H^i (du *topos* C). On voit ainsi à quelle condition un foncteur $C \longrightarrow C'$ induit un homotopisme $SC \longrightarrow SC'$: en vertu du critère cohomologique de Artin-Mazur, il f et s que pour tout système de coefficients F' sur C', l'homomorphisme naturel $\varprojlim^{(i)}_{C'} F' \longrightarrow \varprojlim^{(i)}_{C} F$ soit un isomorphisme (pour les i pour lesquels cela a un sens).

A C on peut associer le topos $\widetilde{C}$, qui varie de façon *covariante* avec C. (NB le foncteur $C \mapsto \widetilde{C}$ n'a plus rien de pleinement fidèle, semble-t-il ??).

Les systèmes de coefficients ensemblistes sur C (les foncteurs $C^\circ \longrightarrow$ Ens transformant isomorphismes en isomorphismes) correspondent aux faisceaux localement constants i.e. les objets localement constants de $\widetilde{C}$, définis intrinsèquement en termes de $\widetilde{C}$. Ainsi, le fait pour un foncteur $F : C \longrightarrow C'$ d'induire une homotopisme $S(C) \longrightarrow S(C')$ ne dépend que du morphisme de topos $\widetilde{F} : \widetilde{C} \longrightarrow \widetilde{C'}$ induit, et signifie que pour tout faisceau localement constant F' sur C' i.e. sur $\widetilde{C'}$, les applications induites $\mathrm{H}^i(\widetilde{C'}, F') \longrightarrow \mathrm{H}^i(\widetilde{C}, \widetilde{F}^*(F'))$ sont des isomorphismes (pour les i pour lesquels cela a un sens).

On a aussi un foncteur évident

$$T : \longrightarrow,$$

en associant à tout ensemble semi-simplicial X la catégorie $T(X) = \Delta_{/X}$ des simplexes sur X, dont l'ensemble des objets est la réunion disjointe des X_n... (c'est une catégorie fibrée sur la catégorie Δ des simplexes types, à fibres les catégories discrètes définies par les X_n). Ceci posé, Quillen prouve que pour tout X, $ST(X)$ est isomorphe canoniquement à X dans la catégorie homotopique construite avec , et que pour toute C, la catégorie $TS(C)$ est canoniquement "homotopiquement équivalente à C" i.e. canoniquement isomorphe a C dans la catégorie quotient de obtenue en inversant les foncteurs qui sont des homotopismes. Ces isomorphismes sont fonctoriels en X. Il en résulte formellement qu'un morphisme $f : X \longrightarrow Y$ dans est un homotopisme si et seulement si en est ainsi de $T(f) : T(X) \longrightarrow T(Y)$, d'où des foncteurs $S' :' \longrightarrow '$ et $T' :' \longrightarrow '$ entre les catégories "homotopiques", construites avec resp , qui sont quasi-inverses l'un de l'autre.

De plus, Quillen contruit un isomorphisme canonique et fonctoriel dans $'$ entre C et la catégorie opposée C°, ou ce qui revient au même, un isomorphisme canonique et fonctoriel dans $'$ entre $S(C)$ et $S(C^\circ)$. La définition est telle que le foncteur induit sur les systèmes locaux sur C transforme le foncteur contravariant F sur C, transformant toute flèche en flèche inversible, en le foncteur covariant (i.e. contravariant sur C°) ayant mêmes valeurs sur les objets, et obtenu sur les flèches en remplaçant $F(u)$ par $F(u)^{-1}$; en d'autres termes, l'effet de l'homotopisme de Quillen sur les groupoïdes fondamentaux est l'isomorphisme évident entre les groupoïdes fondamentaux de C et de C°, compte tenu que le deuxième est l'opposé du premier. Comme application, Quillen obtient une interprétation faisceautique de la cohomologie d'un ensemble semi-simplicial à coefficients dans un système local covariant F (défini classiquement par le complexe cosimplicial des $C^n(F) = \amalg_{x \in X_n} F(x)$): on considère le système local contravariant défini par F, on l'interprète comme un faisceau sur $T(X)$ i.e. objet de $_{/X}$, et on prend sa cohomologie. - Cependant, quand F est un système de coefficients covariant pas nécessairement local, on n'a toujours pas d'interprétation de ses groupes de cohomologie classiques en termes faisceautiques; ni, lorsque F est contravariant, de son homologie, ou inversement de sa cohomologie faisceautique en termes classiques.

A propos de la notion de foncteur qui est un homotopisme. Quillen montre qu'un tel foncteur $F : C \longrightarrow C'$ induit une équivalence entre la sous-catégorie triangulée $\mathbb{D}^b_{lc}(C')$ de la catégorie dérivée bornée de celle des faisceaux abéliens sur C', dont les faisceaux de cohomologie sont des systèmes locaux, et la catégorie analogue pour C; et réciproquement. On peut dans cet énoncé introduire aussi n'importe quel anneau de base (à condition de le supposer $\neq 0$ dans le cas de la réciproque); la partie dire vaut aussi avec un anneau de coefficients par nécessairement constant, mains constant tordu. Je pense que ce résultat (facile) doit pouvoir se généraliser ainsi.

Soit $f : X \longrightarrow X'$ un morphisme de topos qui soit tel que pour tout faisceau localement constant sur X', f induise un isomorphisme sur les cohomologies (avec cas non commutatif inclus). Supposons que X et X' soit *localement homotopiquement trivial*, i.e. que pour tout entier $n \geq 1$, tout objet U ait un recouvrement par des $U_i \longrightarrow U$, tels que a) tout système local sur U devient constant sur U_i, et toute section sur U devient constant sur U_i et b) pour tout groupe abélien G, les $\mathrm{H}^j(U,G) \longrightarrow \mathrm{H}^j(U_i,G)$ sont nuls pour $1 \leq j \leq n$ [2]. Alors le foncteur $\mathbb{D}^b_{lc}(X') \longrightarrow \mathbb{D}^b_{lc}(X)$ induit par f est une équivalence. Même énoncé si on met dans le coup un système local d'anneaux sur X'. Enfin, f induit une équivalence entre la catégorie des coefficients locaux sur C et celle des coefficients locaux sur C'.

n-catégories, catégories n-uples, et Gr-catégories

Point de vue “motivique” en théorie du cobordisme

[2]Attention, cette condition n'est typiquement pas satisfaite par les schémas sur []

Standard Conjectures on Algebraic Cycles

0.8 Introduction

We state two conjectures on algebraic cycles, which arose fro an attempt at understanding the conjectures of Weil on the ζ-functions of algebraic varieties. These are not really new, and they were worked out about three years ago independently by Bombieri and myself.

The first is an existence assertion for algebraic cycles (considerably weaker than the Tate conjectures), and is inspired by and formally analogous to Lefschetz's structure theorem on the cohomology of a smooth projective variety over the complex field.

The second is a statement of positivity, generalising Weil's well-known positivity theorem in the theory of abelian varieties. It is formally analogous to the famous Hodge inequalities, and is in fact a consequence of these in characteristic zero.

WHAT REMAINS TO BE PROVED OF WEIL'S CONJECTURES? Before stating our conjectures, let us recall what remains to be proved in respect of the Weil conjectures, when approached through ℓ-adic cohomology.

Let $X/\mathbf{F}_q$ be a smooth irreducible projective variety of dimension n over the finite field $\overline{\mathbf{F}}_q$ with q elements, and ℓ a prime different from the characteristic. It has then been proved by M. Artin and myself that the Z-function of X can be expressed as

$$Z(t) = \frac{L'(t)}{L(t)},$$
$$L(t) = \frac{L_0(t)L_2(t)\dots L_{2n}(t)}{L_1(t)L_3(t)\dots L_{2n-1}(t)},$$
$$L_i(t) = \frac{1}{P_i(t)},$$

where $P_i(t) = t^{\dim \mathrm{H}^i(\overline{X})} Q_i(t^{-1})$, Q_i being the characteristic polynomial of the action of the Frobenius endomorphism of X on $\mathrm{H}^i(\overline{X})$ (here H^i stands for the i^{th} ℓ-adic cohomology group and $\overline{X}$ is deduced from X by base extension to the algebraic closure of $\mathbf{F}_q$). But it has not been proved so far that

(a) the $P_i(t)$ have integral coefficients, independent of $\ell(\neq \text{char}\,\mathbf{F})$;

(b) the eigenvalues of the Forbenius endomorhisms on $\mathrm{H}^i(\overline{X})$, i.e., the reciprocals of the roots of $P_i(t)$, are of absolute value $q^{i/2}$.

Our first conjecture meets question (a). The first and second together would, by an idea essentially due to Serre [**?**], imply (b).

0.9 A weak form of conjecture 1

From now on, we work with varieties over a ground field k which is algebraically closed and of arbitrary characteristic. Then (a) leads to the following question: If f is an endomorphism of a variety X/k and $\ell \neq \text{char}\, k$, f induces

$$f^i : \mathrm{H}^i(X) \longrightarrow \mathrm{H}^i(x),$$

and each of these f^i has a characteristic polynomial. *Are the coefficients of these polynomials rational integers, and are they independent of ℓ?* When X is smooth and proper of dimension n, the same question is meaningful when f is replaced by any cycle of dimension n in $X \times X$, considered as an algebraic correspondence.

In characteristic zero, one sees that this is so by using integral cohomology. If char $k > 0$, one feels certain that this is so, but this has not been proved so far.

Let us fix for simplicity an isomorphism

$$\ell^{\infty k^* \simeq \mathbf{Q}_\ell/\mathbf{Z}_\ell} \qquad \text{(a heresy!)}.$$

We than have a map

$$: F^i(X) \otimes_{\mathbf{Z}} \mathbf{Q} \longrightarrow \mathrm{H}^{2i}_\ell(X)$$

which associates to an algebraic cycle its cohomology class. We denote by $C^i_\ell(X)$, and refer to its elements as *algebraic cohomology classes*.

A known result, due to Dwork-Faton, shows that for the integrality question (not to speak of the independence of the characteristic polynomial of ℓ), it suffices to prove that

$$f_i^N \in \frac{1}{m}\mathbf{Z} \quad \text{for every } N \geq 0,$$

where m is a fixed positive integer[3]. Now, the graph Γ_{f^N} in $X \times X$ of f^N defines a cohomology class on $X \times X$, and if the cohomology class Δ of the diagonal in $X \times X$ is written as

$$\Delta = \sum_0^n \pi_i$$

where π_i are the projections of Δ onto $\mathrm{H}^i(X) \otimes \mathrm{H}^{n-i}(X)$ for the canonical decomposition $\mathrm{H}^n(X \times X) \simeq \sum_{i=0}^n \mathrm{H}^i(X) \otimes \mathrm{H}^{n-i}(X)$, a known calculation shows that

$$(f^N)_{\mathrm{H}^i} = (-1)^i (\Gamma_{f^N})\pi_i \in \mathrm{H}^{4n}(X \times X) \approx \mathbf{Q}_\ell.$$

Assume that the π_i are algebraic. Then $\pi_i = \frac{1}{m}(\prod_i)$, where $\prod_i$ is an algebraic cycle, hence

$$(f^N)_{\mathrm{H}^i} = (-1)^i (\prod_i \Gamma_{f^N}) \in \frac{1}{m}\mathbf{Z}$$

and we are through.

WEAK FORM OF CONJECTURE 1. $(C(X))$: The elements π_i^ℓ are algebraic, (and come from an element of $F^i(X) \otimes_{\mathbf{Z}} \mathbf{Q}$, which is independent of ℓ).

N.B.

1. The statement in parenthesis is needed to establish the independence of P_i on ℓ.

2. If $C(X)$ and $C(Y)$ hold, $C(X \times Y)$ holds, and more generally, the Künneth components of any algebraic cohomology class on $X \times Y$ are algebraic.

[3]This was pointed out to me by S. Kleimann.

0.10 The conjecture 1 (of Lefschetz type)

Let X be smooth and projective, and $\xi \in \mathrm{H}^2(X)$ the class of a hyperplane section. Then we have a homomorphism

$$(*) \qquad \cup\xi^{n-i} : \mathrm{H}^i(X) \longrightarrow \mathrm{H}^{2n-i}(X) \quad (i \leq n).$$

It is expected (and has been established by Lefschetz [?], [?] over the complex field by transcendental methods) that this is an isomorphism for all characteristics. For $i = 2j$, we have the commutative square

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Our conjecture is then: $(A(X))$:

(a) *(*) is always an isomorphism (the mild form);*

(b) *if $i = 2j$. (*) induces an isomorphism (or equivalently, an epimorphism) $C^j(X) \longrightarrow C^{n-j}(X)$.*

N.B. If $C^j(X)$ is assumed to be finite dimensional, (b) is equivalent to the assertion that $\dim C^{n-j}(X) \leq \dim C^j(X)$ (which in particular implies the equality of these dimensions in view of (a)).

An equivalent formulation of the above conjecture (for all varieties X as above) is the following.

$(B(X))$: *The Λ-operation (c.f. [?]) of Hodge theory is algebraic.*

By this, we mean that there is an algebraic cohomology class λ in $\mathrm{H}^*(X \times X)$ such that the map $\Lambda : \mathrm{H}^*(X) \longrightarrow \mathrm{H}^*(X)$ is got by lifting a class from X to $X \times X$ by the first projection, cupping with λ and taking the image in $\mathrm{H}^*(X)$ by the Gysin homomorphism associated to the second projection

Note that $B(X) \Rightarrow A(X)$, since the algebricity of λ implies that of λ^{n-i}, and λ^{n-i} provides an inverse to $\cup\xi^{n-i} : \mathrm{H}^i(X) \longrightarrow \mathrm{H}^{2n-i}(X)$. On the other hand, it is easy to show that $A(X \times X) \Rightarrow B(X)$ and this proves the equivalence of conjectures A and B.

The conjecture seems to be most amenable in the form of B. Note that $B(X)$ is stable for products, hyperplane sections and specialisations. In particular, since it holds for projective spaces, it is also true or smooth varieties which are complete intersections in some projective space. (As a consequence, we deduce for such varieties the wished-for integrality theorem for the Z-function!). It is also verified for Grassmannians, and for abelian varieties (Liebermann [?]).

I have an idea of a possible approach to Conjecture B, which relies in turn on certain unsolved geometric questions, and which should be settled in any case.

Finally, we have the implication $B(X) \Rightarrow C(X)$ (first part), since the π_i can be expressed as polynomials with coefficients in $\mathbf{Q}$ of λ and $L = \cup\xi$. To get the whole of $C(X)$, one should naturally assume further that there is an element of $F(X \times X) \otimes_{\mathbf{Z}} \mathbf{Q}$ which gives λ for every ℓ.

0.11 Conjecture 2 (of Hodge type)

For any $i \leq n$, let $P^i(X)$ be the 'primitive part' of $\mathrm{H}^i(X)$, that is, the kernel of $\cup\xi^{n-i+1}$: $\mathrm{H}^i(X) \longrightarrow \mathrm{H}^{2n-i+2}(X)$, and put $C^j_{Pr}(X) = P^{2j} \cap C^j(X)$. On $C\Theta\widehat{\ }_{Pr}(X)$, we have a $\mathbf{Q}$-valued symmetric bilinear form given by

$$(x, y) \longrightarrow (-1)^j K(xy\xi^{n-2j})$$

where K stands for the isomorphism $\mathrm{H}^{2n}(X) \simeq \mathbf{Q}_\ell$. Our conjecture is then that

$((X))$: *The above form is positive definite.*

One is easily reduced to the case when $\dim X = 2m$ is even, and $j = m$.

REMARKS.

(1) In characteristic zero, this follows readily from Hodge theory [?].

(2) $B(X)$ and $Hdg(X \times X)$ imply, by certain arguments of Weil and Serre, the following: if f is an endomorphism of X such that $f^*(\xi) = q\xi$ for some $q \in \mathbf{Q}$ (which is necessarily > 0), then the eigenvalues of $f_{\mathrm{H}^i(X)}$ are algebraic integers of absolute value $q^{i/2}$. Thus, this implies all of Weil's conjectures.

(3) The conjecture $Hdg(X)$ together with $A(X)(a)$ (the Lefschetz conjecture in cohomology) implies that numerical equivalence of cycles is the same as cohomological equivalence for any ℓ-adic cohomology if and only if $A(X)$ holds.

(4) In view of (3), $B(X)$ and $Hdg(X)$ imply that numerical equivalence of cycles coincides wit $\mathbf{Q}_\ell$-equivalence for any ℓ. Further the natural map

$$\mathrm{Z}^i(X) \otimes_{\mathbf{Z}} \mathbf{Q}_\ell \longrightarrow \mathrm{H}^i_\ell(X)$$

is a monomorphism, and in particular, we have

$$\dim_{\mathbf{Q}} C^i(X) \leq \dim_{\mathbf{Q}_\ell} \mathrm{H}^i_\ell(X).$$

Note that for the deduction of this, we do not make use of the positivity of the form considered in (X),but only the fact that it is non-degenerate.

Another consequence of $Hdg(X)$ and $B(X)$ is that the stronger version of $B(X)$, viz. that λ comes from an algebraic cycle with rational coefficients *independent of* ℓ, holds.

0.12 Conclusions

The proof of the two standard conjectures would yield results going considerably further than Weil's conjectures. They would form the basis of the so-called "theory of motives" which is a systematic theory of "arithmetic properties" of algebraic varieties, as embodied in their groups of classes of cycles for numerical equivalence. We have at present only a very small part of this theory in dimension one, as contained in the theory of abelian varieties.

Alongside the problem of resolution of singularities, the proof of the standard conjectures seems to me to be the most urgent task in algebraic geometry.

Esquisse d'un Programme

LochakSchneps2000

0.1 Envoi

Comme la conjoncture actuelle rend de plus en plus illusoire pour moi les perspectives d'un enseignement de recherche à l'Université, je me suis résolu à demander mon admission au CNRS, pour pouvoir consacrer mon énergie à développer es travaux et perspectives dont il devient clair qu'il ne se trouvera aucun élève (ni même, semble-t-il, aucun congénère mathématicien) pour les développer à ma place.

En guise de document "Titres et Travaux", on trouvera à la suite de ce texte la reproduction intégrale d'une esquisse, par thèmes, de ce que je considérais comme mes principales contributions mathématiques au moment d'écrire ce rapport, en 1972. Il contient également une liste d'articles publiés à cette date. J'ai cessé toute publication d'articles scientifiques depuis 1970. Dans les lignes qui suivent, je me propose de donner un aperçu au moins sur quelques thèmes principaux de mes réflexions mathématiques depuis lors. Ces réflexions se sont matérialisées au cours des années en deux volumineux cartons de notes manuscrites, difficilement déchiffrables sans doute à tout autre qu'à moi-même, et qui, après des stades de décantations successives, attendent leur heure peut-être pour une rédaction d'ensemble tout au moins provisoire, à l'intention de la communauté mathématique. Le terme "rédaction" ici est quelque peu impropre, alors qu'il s'agit bien plus de développer des idées et visions multiples amorcées au cours de ces douze dernières années, en les précisant et les approfondissant, avec tous les rebondissements imprévus qui constamment accompagnent ce genre de travail – un travail de découverte donc, et non de compilation de notes pieusement accumulées. Et je compte bien, dans l'écriture des "Réflexions Mathématiques" commencée depuis février 1983, laisser apparaître clairement au fil des pages la démarche de la pensée qui sonde et qui découvre, en tâtonnant dans la pénombre bien souvent, avec des trouées de lumière subites quand quelque tenace image fausse, ou simplement inadéquate, se trouve enfin débusquée et mise à jour, et que les choses qui semblaient de guingois se mettent en place, dans l'harmonie mutuelle qui leur est propre.

Quoi qu'il en soit, l'esquisse qui suit de quelques thèmes de réflexions des dernières dix ou douze années, tiendra lieu en même temps d'esquisse de programme de travail pour les années qui viennent, que je compte consacrer au développement de ces thèmes, ou au moins de certains d'entre eux. Elle est destinée, d'une part aux collègues du Comité National appelés à statuer sur ma demande, d'autre part à quelques autres collègues, anciens élèves, amis, dans l'éventualité où certaines des idées esquissées ici pourraient intéresser l'un d'entre eux.

0.2 Un jeu de "Lego-Teichmüller" et le groupe de Galois de $\overline{\mathbf{Q}}$ sur $\mathbf{Q}$

0.3 Corps de nombres associés à un dessin d'enfant

0.4 Polyèdres réguliers sur les corps finis

0.5 Haro sur la topologie dite "générale", et réflexions heuristiques vers une topologie dite "modérée"

0.6 “Théories différentielles” (à la Nash) et “théories modérées”

0.7 À la Poursuite des Champs

0.8 Digressions de géométrie bidimensionnelle

8. . –

0.9 Bilan d’une activité enseignante

0.10 Épilogue

Sketch of a Programme

0.1 Preface

As the present situation makes the prospect of teaching at the research level at the University seem more and more illusory, I have resolved to apply for admission to the CNRS, in order to devote my energy to the development of projects and perspectives for which it is becoming clear that no student (nor even, it seems, any mathematical colleague) will be found to develop them in my stead.

In the role of the document "Titles and Articles", one can find after this text the complete reproduction of a sketch, by themes, of what I considered to be my principal mathematical contributions at the time of writing that report, in 1972. It also contains a list of articles published at that date. I ceased all publication of scientific articles in 1970. In the following lines, I propose to give a view of at least some of the principal themes of my mathematical reflections since then. These reflections materialised over the years in the form of two voluminous boxes of handwritten notes, doubtless difficult to decipher for anyone but myself, and which, after several successive stages of settling, are perhaps waiting for their moment to be written up together at least in a temporary fashion, for the benefit of the mathematical community. The term "written up" is somewhat incorrect here, since in fact it is much more a question of developing the ideas and the multiple visions begun during these last twelve years, to make them more precise and deeper, with all the unexpected rebounds which constantly accompany this kind of work – a work of discovery, thus, and not of compilation of piously accumulated notes. And in writing the "Mathematical Reflections", begun since February 1983, I do intend throughout its pages to clearly reveal the process of thought, which feels and discovers, often blindly in the shadows, with sudden flashes of light when some tenacious false or simply inadequate image is finally shown for what it is, and things which seemed all crooked fall into place, with that mutual harmony which is their own.

In any case, the following sketch of some themes of reflection from the last ten or twelve years will also serve as a sketch of my programme of work for the coming years, which I intend to devote to the development of these themes, or at least some of them. It is intended on the one hand for my colleagues of the National Committee whose job it is to decide the fate of my application, and on the other hand for some other colleagues, former students, friends, in the possibility that some of the ideas sketched here might interest one of them.

0.2 A game of "Lego-Teichmüller" and the Galois group $\overline{\mathbf{Q}}$ over $\mathbf{Q}$

The demands of university teaching, addressed to students (including those said to be "advanced") with a modest (and frequently less than modest) mathematical baggage, led me to a Draconian renewal of the themes of reflection I proposed to my students, and gradually to myself as well. It seemed important to me to start from an intuitive baggage common to everyone, independent of any technical language used to express it, and anterior to any such language – it turned out that the geometric and topological intuition of shapes, particularly two-dimensional shapes, formed such a common ground. This consists of themes which can be grouped under the general name of "topology of surfaces" or "geometry of surfaces", it being understood in this last expression that the main emphasis is on the topological properties of the surfaces, or the combinatorial aspects which form the most down-to-earth technical expression of them, and not on the differential, conformal, Riemannian, holomorphic aspects, and (from there) on to "complex algebraic curves". Once this last step is taken, however, algebraic geometry (my former love!) suddenly bursts forth once again, and this via the objects which we can consider as the basic building blocks for all other algebraic varieties. Whereas in my research before 1970, my attention was systematically directed towards objects of maximal generality, in order to uncover a general language adequate for the world of algebraic geometry, and I never restricted myself to algebraic curves except when strictly necessary (notably in étale cohomology), preferring to develop "pass-key" techniques and statements valid in all dimensions and in every place (I mean, over all base schemes, or even base ringed topoi…), here I was brought back, via objects so simple that a child learns them while playing, to the beginnings and origins of algebraic geometry, familiar to Riemann and his followers!

Since around 1975, it is thus the geometry of (real) surfaces, and starting in 1977 the links between questions of geometry of surfaces and the algebraic geometry of algebraic curves defined over fields such as **C**, **R** or extensions of **Q** of finite type, which were my principal source of inspiration and my constant guiding thread. It is with surprise and wonderment that over the years I discovered (or rather, doubtless, rediscovered) the prodigious, truly inexhaustible richness, the unsuspected depth of this theme, apparently so anodine. I believe I feel a central sensitive point there, a privileged point of convergence of the principal currents of mathematical ideas, and also of the principal structures and visions of things which they express, from the most specific (such as the rings $\mathbf{Z}$, $\mathbf{Q}$, $\overline{\mathbf{Q}}$, $\mathbf{R}$, $\mathbf{C}$ or the group (2) over one of these rings, or general reductive algebraic groups) to the most "abstract", such as the algebraic "multiplicities", complex analytic or real analytic. (These are naturally introduced when systematically studying "moduli varieties" for the geometric objects considered, and if we want

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0.3 Number fields associated to a child's drawing

Instead of following (as I meant to) a rigorous thematic order, I let myself be carried away by my predilection for a particularly rich and burning theme, to which I intend to devote myself prioritarily for some time, starting at the beginning of the academic year 84/85. Thus I will take the thematic description up again where I left it, at the very beginning of the preceding paragraph.

My interest in topological surfaces began to appear in 1974, when I proposed to Yves Ladegaillerie the theme of the isotopic study of embeddings of a topological 1-complex into a compact surface. Over the two following years, this study led him to a remarkable isotopy theorem, giving a complete algebraic description of the isotopy classes of embeddings of such

1-complexes, or compact surfaces with boundary, in a compact oriented surface, in terms of certain very simple combinatorial invariants, and the fundamental groups of the protagonists. This theorem, which should be easily generalisable to embeddings of any compact space (triangulable to simplify) in a compact oriented surface, gives as easy corollaries several deep classical results in the theory of surfaces, and in particular Baer's isotopy theorem. It will finally be published, separately from the rest (and ten years later, seeing the difficulty of the times…), in Topology. In the work of Ladegaillerie there is also a purely algebraic description, in terms of fundamental groups, of the "isotopic" category of compact surfaces X, equipped with a topological 1-complex K embedded in X. This description, which had the misfortune to run counter to "today's taste" and because of this appears to be unpublishable, nevertheless served (and still serves) as a precious guide in my later reflections, particularly in the context of absolute algebraic geometry in characteristic zero.

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0.4 Regular polyhedra over finite fields

From the very start of my reflection on 2-dimensional maps, I was most particularly interested by the "regular" maps, those whose automorphism group acts transitively (and consequently, simply transitively) on the set of flags. In the oriented case and in terms of the algebraic-geometric interpretation given in the preceding paragraph, it is these maps which correspond to Galois coverings of the projective line.

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0.5 Denunciation of so-called "general" topology, and heuristic reflections towards a so-called "tame" topology

I would like to say a few words now about some topological considerations which have made me understand the necessity of new foundations for "geometric" topology, in a direction quite different from the notion of topos, and actually independent of the needs of so-called "abstract" algebraic geometry (over general base fields and rings). The problem I started from, which already began to intrigue me some fifteen years ago, was that of defining a theory of "dévissage" for stratified structures, in order to rebuild them, via a canonical process, out of "building blocks" canonically deduced from the original structure. Probably the main example which had led me to that question was that of the canonical stratification of a singular algebraic variety (or a complex or real singular space) through the decreasing sequence of its successive singular loci. But I probably had the premonition of the ubiquity of stratified structures in practically all domains of geometry (which surely others had seen clearly a long time before). Since then, I have seen such structures appear, in particular, in any situation where "moduli" are involved for geometric objects which may undergo not only continuous variations, but also "degeneration" (or "specialisation") phenomena – the strata corresponding then to the various "levels of singularity" (or to the associated combinatorial types) for the objects in question. The compactified modular multiplicities $\widehat{M}_{g,\nu}$ of Mumford-Deligne for the stable algebraic curves of type (g,ν) provide a typical and particularly inspiring example, which played an important motivating role when I returned to my reflection about stratified structures, from December 1981 to January 1982. Two-dimensional geometry provides many other examples of such modular stratified structures, which all (if not using rigidification) appear as "multiplicities" rather than as spaces or manifolds in the usual sense (as the points of these multiplicities may have non-trivial automorphism groups). Among the objects of two-dimensional geometry which give rise to such modular stratified structures in arbitrary dimensions, or even infinite dimension, I would list polygons (Euclidean, spherical or hyperbolic), systems of straight lines in a plane (say projective), systems of "pseudo straight lines" in a projective topological plane, or more general immersed curves with normal crossings, in a given (say compact) surface.

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0.6 “Differentiable theories” (à la Nash) and “tame theories”

One of the most interesting foundational theorems of (tame) topology which should be developed would be a theorem of “dévissage” (again!) of a proper tame map of tame spaces

$$f : X \longrightarrow Y,$$

via a

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0.7 Pursuing Stacks

Since the month of March last year, so nearly a year ago, the greater part of my energy has been devoted to a work of reflection on the *foundations of non-commutative (co)homological algebra*, or what is the same, after all, of *homotopical algebra*. These reflections have taken the concrete form of a voluminous stack of typed notes, destined to form the first volume (now being finished) of a work in two volumes to be published by Hermann, under the overall title *“Pursuing Stacks”*. I now foresee (after successive extensions of the initial project) that the manuscript of the whole of the two volumes, which I hope to finish definitively in the course of this year, will be about 1500 typed pages in length. These two volumes are moreover for me the first in a vaster series, under the overall title *“Mathematical Reflections”*, in which I intend to develop some of the themes sketched in the present report.

Since I am speaking here of work which is actually now being written up and is even almost finished, the first volume of which will doubtless appear this year and will contain a detailed introduction, it is undoubtedly less interesting for me to develop this theme of reflection here, and I will content myself with speaking of it only very briefly. This work seems to me to be somewhat marginal with respect to the themes I sketched before, and does not (it seems to me) represent a real renewal of viewpoint or approach with respect to my interests and my mathematical vision of before 1970. If I suddenly resolved to do it, it is almost out of desperation, for nearly twenty years have gone by since certain visibly fundamental questions, which were ripe to be thoroughly investigated, without anyone seeing them or taking the trouble to fathom them. Still today, the basic structures which occur in the homotopical point of view in topology are not understood, and to my knowledge, after the work of Verdier, Giraud and Illusie on this theme (which are so many beginnings still waiting for continuations…) there has been no effort in this direction. I should probably make an exception for the axiomatisation work done by Quillen on the notion of a category of models, at the end of the sixties, and taken up in various forms by various authors. At that time, and still now, this work seduced me and taught me a great deal, even while going in quite a different direction from the one which was and still is close to my heart. Certainly, it introduces derived categories in various non-commutative contexts, but without entering into the question of the essential internal structures of such a category, also left open in the commutative case by Verdier, and after him by Illusie. Similarly, the question of putting one's finger on the natural "coefficients" for a non-commutative cohomological formalism, beyond the stacks (which should be called 1-stacks) studied in the book by Giraud, remained open – or rather, the rich and precise intuitions concerning it, taken from the numerous examples coming in particular from algebraic geometry, are still waiting for a precise and supple language to give them form.

I returned to certain aspects of these foundational questions in 1975, on the occasion (I seem to remember) of a correspondence with Larry Breen (two letters from this correspondence will be reproduced as an appendix to Chap. I of volume 1, "History of Models", of Pursuing Stacks). At that moment the intuition appeared that ∞-groupoids should constitute particularly adequate models for homotopy types, the n-groupoids corresponding to *truncated* homotopy types (with $\pi_i = 0$ pour $i > n$). This same intuition, via very different routes, was discovered by Ronnie Brown and some of his students in Bangor, but using a rather restrictive notion of ∞-groupoid (which, among the 1-connected homotopy types, model only products of Eilenberg-Mac Lane spaces). Stimulated by a rather haphazard correspondence with Ronnie Brown, I finally began this reflection, starting with an attempt to define a wider notion of ∞-groupoid (later rebaptised stack in ∞-groupoids or simply "stack", the implication being: over the 1-point topos), and which, from one thing to another, led me to Pursuing Stacks. The volume "History of Models" is actually a completely unintended digression with respect to the initial project (the famous stacks being temporarily forgotten, and supposed to reappear only around page 1000…).

This work is not completely isolated with respect to my more recent interests. For example, my reflection on the modular multiplicities $\widehat{M}_{g,\nu}$ and their stratified structure renewed the reflection on a theorem of van Kampen in dimension > 1 (also one of the preferred themes of the group in Bangor), and perhaps also contributed to preparing the ground for the more important work of the following year. This also links up from time to time with a reflection dating from the same year 1975 (or the following year) on a "De Rham complex with divided powers", which was the subject of my last public lecture, at the IHES in 1976; I lent the manuscript of it to I don't remember whom after the talk, and it is now lost. It was at the moment of this reflection that the intuition of a "schematisation" of homotopy types germinated, and seven years later I am trying to make it precise in a (particularly hypothetical) chapter of the History of Models.

The work of reflection undertaken in Pursuing Stacks is a little like a debt which I am paying towards a scientific past where, for about fifteen years (from 1955 to 1970), the development of cohomological tools was the constant Leitmotiv in my foundational work on algebraic geometry. If in this renewal of my interest in this theme, it has taken on unexpected dimensions, it is however not out of pity for a past, but because of the numerous unexpected phenomena which ceaselessly appear and unceremoniously shatter the previously laid plans and projects – rather like in the thousand and one nights, where one awaits with bated breath through twenty other tales the final end of the first.

0.8 Digressions on 2-dimensional geometry

Up to now I have spoken very little of the more down-to-earth reflections on two-dimensional topological geometry, directly associated to my activities of teaching and "directing research". Several times, I saw opening before me vast and rich fields ripe for the harvest, without ever succeeding in communicating this vision, and the spark which accompanies it, to one of my students, and having it open out into a more or less long-term common exploration. Each time up through today, after a few days or weeks of investigating where I, as scout, discovered riches at first unsuspected, the voyage suddenly stopped, upon its becoming clear that I would be pursuing it alone. Stronger interests then took precedence over a voyage which at that point appeared more as a digression or even a dispersion, than a common adventure.

One of these themes was that of planar polygons, centred around the modular varieties which can be associated to them. One of the surprises here was the irruption of algebraic geometry in a context which had seemed to me quite distant. This kind of surprise, linked to the omnipresence of algebraic geometry in plain geometry, occurred several times.

0.9 Assessment of a teaching activity

The occasion appears to be auspicious for a brief assessment of my teaching activity since 1970, that is, since it has taken place in a university. This contact with a very different reality taught me many things, of a completely different order than simply pedagogic or scientific. Here is not the place to dwell on this subject. I also mentioned at the beginning of this report the role which this change of professional milieu played in the renewal of my approach to mathematics, and that of my centres of interest in mathematics. If I pursue this assessment of my teaching activity on the research level, I come to the conclusion of a clear and solid failure. In the more than ten years that this activity has taken place, year after year in the same university, I was never at any moment able to suscitate a place where "something happened" – where something "passed", even among the smallest group of people, linked together by a common adventure. Twice, it is true, around the years 1974 to 1976, I had the pleasure and the privilege of awakening a student to a work of some consequence, pursued with enthusiasm: Yves Ladegaillerie in the work mentioned earlier (par. 3) on questions of isotopy in dimension 2, and Carlos Contou-Carrére (whose mathematical passion did not await a meeting with myself to blossom) an unpublished work on the local and global Jacobians over general base schemes (of which one part was announced in a note in the CR). Apart from these two cases, my role has been limited throughout these ten years to somehow or other conveying the rudiments of the mathematician's trade to about twenty students on the research level, or at least to those among them who persevered with me, reputed to be more demanding than others, long enough to arrive at a first acceptable work written black on white (and even, sometimes, at something better than acceptable and more than just one, done with pleasure and worked out through to the end). Given the circumstances, among the rare people who persevered, even rarer are those who will have the chance of carrying on the trade, and thus, while earning their bread, learning it ever more deeply.

0.10 Epilogue

Since last year, I feel that as regards my teaching activity at the university, I have learned everything I have to learn and taught everything I can teach there, and that it has ceased to be really useful, to myself and to others. To insist on continuing it under these circumstances would appear to me to be a waste both of human resources and of public funds. This is why I have applied for a position in the CNRS (which I left in 1959 as freshly named director of research, to enter the IHES). I know moreover that the employment situation is tight in the CNRS as everywhere else, that the result of my request is doubtful, and that if a position were to be attributed to me, it would be at the expense of a younger researcher who would remain without a position. But it is also true that it would free my position at the USTL to the benefit of someone else. This is why I do not scruple to make this request, and to renew it if is not accepted this year.

In any case, this application will have been the occasion for me to write this sketch of a programme, which otherwise would probably never have seen the light of day. I have tried to be brief without being sibylline and also, afterwards, to make it easier reading by the addition of a summary. If in spite of this it still appears rather long for the circumstances, I beg to be excused. It seems short to me for its content, knowing that ten years of work would not be too much to explore even the least of the themes sketched here through to the end (assuming that there is an "end"…), and one hundred years would be little for the richest among them!

Behind the apparent disparity of the themes evoked here, an attentive reader will perceive as I do a profound unity. This manifests itself particularly by a common source of inspiration, namely the geometry of surfaces, present in all of these themes, and most often front and centre. This source, with respect to my mathematical "past", represents a renewal, but certainly not a rupture. Rather, it indicates the path to a new approach to the still mysterious reality of *"motives"*, which fascinated me more than any other in the last years of this past[4]. This fascination has certainly not vanished, rather it is a part of the fascination with the most burning of all the themes evoked above. But today I am no longer, as I used to be, the voluntary prisoner of interminable tasks, which so often prevented me from springing into the unknown, mathematical or not. The time of tasks is over for me. If age has brought me something, it is lightness.

[4]On this subject, see my commentaries in the "Thematic Sketch" of 1972 attached to the present report, in the last section "motivic disgressions", (loc. cit. pages 17-18)