Paris Feb 5, 1962

Dear John,[1]

In connection with my Bourbaki talk[2], I pondered again on Picard schemes. For instance, as I told Mumford, I proved that if X/S is projective and simple,[3] then $\mathcal{P}ic^{\tau}_{X/S}$ is of finite type over S. More generally, the decomposition of $\mathcal{P}ic_{X/S}$ according to the Hilbert polynomials (in fact, the first two non trivial coefficients of the polynomial suffice) consists of pieces which are of finite type, hence projective over S. Another way of stating this is to say that a family of divisors D_i on the geometric fibers of X/S is "limited" iff the projective degrees of the D_i and D_i^2 are bounded.

Another result, of interest in connection with your seminar, is a proof of the fact that, for an abelian scheme A/k, k a perfect field, the absolute formal scheme of moduli over $\mathbb{W}_\infty(k)$ is simple over k. This comes from the following general fact: Let X_0/S_0 be simple, X_0'/X_0 étale, S_0 subscheme of S defined by an ideal $\mathcal{I}$ of square 0. Let $\xi_0 \in H^2(X_0, \mathfrak{G}_{X_0/S_0} \otimes_{\mathcal{O}_{S_0}} \mathcal{I})$ and[4] $\xi_0' \in H^2(X_0', \mathfrak{G}_{X_0'/S_0} \otimes_{\mathcal{O}_{S_0}} \mathcal{I})$ be the obstruction for lifting. Then ξ_0' is the inverse image of ξ_0 under the obvious map. As a consequence, if X_0/S_0 is abelian, taking $X_0' = X_0$, $X_0' \to X_0$ multiplication by n prime to the residue characteristic, we get $\xi_0 = n^*(\xi_0)$. If $S = \operatorname{Spec} \Lambda$, Λ local artin, and $\mathfrak{m}\,\mathcal{I} = 0$, then we are reduced to an obstruction in the H^2 of the reduced $X_0 \otimes_{\Lambda_0} k = A$, satisfying $\xi = n^*(\xi)$ for n prime to p. Using the structure

$$H^*(A, \mathfrak{G}_{A/k}) \simeq \bigwedge^{*} H^1(A, \mathcal{O}_A) \otimes t_A\,,$$

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[1] Letter to John Tate.

[2] Referring to Séminaire Bourbaki 1960/61, n°232 and n°236, V. Les schémas de Picard. Théorèmes d'existence. VI. Les schémas de Picard. Propriétés générales.

[3] The standard terminology has changed from "simple" to "smooth".

[4] Here $\mathfrak{G}_{X_0/S_0}$ and $\mathfrak{G}_{X_0'/S_0}$ denote the relative tangent sheaves for X_0/S_0 and X_0'/S_0 respectively.

we get $n^*(\xi) = n^3\xi$, hence $(n^3-1)\xi = 0$. Taking $n = -1$ we get $2\xi = 0$, hence $\xi = 0$, and we win!

I just noticed[5] the proof does not give any information for residue char. $=$ 2! Here is a simple proof valid in any char.: Consider the obstruction η_0 for lifting $X_0 \times_{S_0} X_0$, then $\eta_0 = \xi_0 \otimes 1 + 1 \otimes \xi_0$, and η_0 is invariant under the *automorphism* $(x, y) \rightsquigarrow (x, y + x)$ of $X_0 \times_{S_0} X_0$. Thus we get an element $\xi = \sum_{i,j} \lambda_{i,j} e_i \wedge e_j$ in $H^2(A, \mathcal{O}_A) = \bigwedge^2 t$, s.th. $\eta = \sum_{i,j} \lambda_{i,j} e'_i \wedge e'_j + \sum_{i,j} \lambda_{i,j} e''_i \wedge e''_j$ in $\bigwedge^2(t \oplus t)$ is *invariant* under $(x, y) \rightsquigarrow (x, y + x)$, carrying $e'_i \rightsquigarrow e'_i + e''_i$ and $e''_i \rightsquigarrow e''_i$, hence trivially $\xi = 0$!

As a consequence, we get that the scheme of moduli for the *polarized* abelian schemes, with polarization degree d, is simple over $\mathbb{Z}$ at all those primes p which do not divide d. This comes from the fact that the obstruction to polarized lifting lies in a module $H^2(A, \mathcal{E})$, where $\mathcal{E}$ is an extension (the "Atiyah extension")

$$(*) \qquad 0 \to \mathcal{O}_A \to \mathcal{E} \to \mathfrak{G}_{A/k} \to 0$$

whose class c in $H^1(A, \Omega^1_{A/k})$ is just the Chern class $\frac{d\mathcal{L}}{\mathcal{L}}$ of the invertible sheaf $\mathcal{L}$ on A defining the polarization. Now in the exact sequence of cohomology for $(*)$, the map

$$\begin{array}{ccc} H^i(\mathfrak{G}_{A/k}) & \xrightarrow{\partial^{(i)}} & H^{i+1}(\mathcal{O}_A) \\ \simeq\downarrow & & \simeq\downarrow \\ \bigwedge^i t' \otimes t & & \bigwedge^{i+1} t' \end{array} \qquad t = t_A, t' = t_{\hat{A}}$$

is trivially described in terms of

$$c \in H^1(A, \Omega^1_{A/k}) \simeq \mathrm{Hom}(t, t'),$$

where the homomorphism $c \colon t \to t'$ is just the tangent map for $\varphi \colon A \to \hat{A}$ defined by the polarization. This map being surjective by assumption, $\partial^{(i)}$ is surjective, hence $H^i(\mathcal{E}) \to H^i(\mathfrak{G}_{A/k})$ is injective, in particular

$$H^2(\mathcal{E}) \to H^2(\mathfrak{G}_{A/k})$$

is *injective*. As the obstructions obtained in $H^2(\mathfrak{G}_{A/k})$ are zero, the same holds for the polarized obstructions in $H^2(\mathcal{E})$, hence the assertion of the simplicity. (If however $p|d$, simplicity *does not hold* at *any* point of $\mathcal{M}$ over p!)

Using the simplicity for the formal scheme of moduli of abelian varieties, I can prove the following:

Let X/Λ be flat, proper, $H^0(X_0, \mathcal{O}_0) \xleftarrow{\sim} k$, where Λ is local artin with residue field k. Assume $\mathcal{P}ic_{X_0/k}$ exists, and is *simple* over k, i.e. $\dim \mathcal{P}ic_{X_0/k} = \dim H^1(X_0, \mathcal{O}_{X_0})$ (always true in char 0). Then

a) $\mathcal{P}ic^0_{X/\Lambda}$ exists and is an *abelian* scheme over Λ.

[5] This paragraph was penned on the left margin vertically.

b) The "base extension property" holds for $R^if_*(\mathcal{O}_X)$ in dimension 1, and more generally in any dimension i such that

$$\bigwedge^i H^1(X_0, \mathcal{O}_{X_0}) \to H^i(X_0, \mathcal{O}_{X_0})$$

is *surjective*, and $H^1(X, \mathcal{O}_X)$ is free over Λ.

Idea of proof:

a) $\mathcal{P}ic^0_{X/k}$ is constructed stepwise. Having $\mathcal{P}ic^0_{X_{n-1}/k} = A_{n-1}$, to get A_n we first lift *arbitrarily* A_{n-1} to an abelian scheme A'_n. We then try to construct the can. invertible "Weil sheaf" on $X_n \times_{\Lambda_n} A'_n$, extending the given Weil sheaf on $X_{n-1} \times_{\Lambda_{n-1}} A_{n-1}$. The obstruction lies in

$$H^2(X_0 \times A_0, \mathcal{O}_{X_0 \times A_0}) \simeq H^2(\mathcal{O}_{X_0}) \times H^2(\mathcal{O}_{A_0}) \times H^1(\mathcal{O}_{X_0}) \otimes H^1(\mathcal{O}_{A_0})$$

and in fact, as easily seen, in the last factor $H^1(X_0, \mathcal{O}_{X_0}) \otimes H^1(A_0, \mathcal{O}_{A_0}) \simeq t_{A_0} \otimes H^1(A_0, \mathcal{O}_{A_0}) \simeq H^1(A_0, \mathfrak{G}_{A_0/k})$. This space is exactly the group operating in a simply transitive way on the set of all extensions of A_{n-1}. Thus we can *correct* A'_n in just one way to get an A_n with a "Weil sheaf" on it! This does it.

b) Let ω be the conormal sheaf to the unit section of $A = \mathcal{P}ic^0_{X/S}$, thus ω is *free* because A/S is simple, and by definition of $\mathcal{P}ic^0_{X/S}$ we have

$$H^1(X, \mathcal{O}_A) \simeq \mathrm{Hom}(\omega, \mathcal{O}_S)$$

This description holds also after any base extension, hence the fact that $H^1(X, \mathcal{O}_X)$ is free over Λ and its formation commutes with base extension. This implies also $H^1(X, \mathcal{O}_X) \to H^1(X_0, \mathcal{O}_{X_0})$ surjective, hence $H^i(X, \mathcal{O}_X) \to H^i(X_0, \mathcal{O}_{X_0})$ is surjective for the i's as in the theorem, ok.

Corollary. Let A/S be any abelian scheme, then the modules $R^if_*(\mathcal{O}_A)$ on S are locally free and in fact $\simeq \bigwedge^i R^1f_*(\mathcal{O}_A)$. If $\mathcal{P}ic_{A/S}$ exists, then $\mathcal{P}ic^0_{A/S}$ is open and is an abelian scheme over S.
(Moreover, biduality holds, as follows easily from the statement over a field ...).
Corollary. Let $f : X \to S$ be flat, proper, $k(s) \xrightarrow{\sim} H^0(X_s, \mathcal{O}_{X_s})$ for every s, let $s \in S$ be such that $\dim H^1(X_s, \mathcal{O}_{X_s}) = \dim \mathcal{P}ic_{X_s/k(s)}$, (the latter defined, if $\mathcal{P}ic_{X_s/k(s)}$ is not known to exist, in terms of the formal Picard scheme). Then $R^1f_*(\mathcal{O}_X)$ is free at s.
This is always applicable if char $k = 0$.

I do not know if, in the case considered, the $R^if_*(\mathcal{O}_X)$ or even $R^if_*(\Omega^j_{X/S})$ are also free at s, even in char 0. It is true for $f_*(\Omega^1_{X/S})$ whenever we know that $\dim H^1(X_s, \mathcal{O}_{X_s}) = \dim H^0(X_s, \Omega^1_{X_s})$, for instance if char $k(s) = 0$ and $f : X \to S$ is projective and simple. (If *moreover* S is reduced, Hodge theory implies *all* $R^if_*(\Omega^j_{X/S})$ are free at s; but if S is artin, I have no idea!)

I now doubt very much that it be true in general that $\mathcal{P}ic^{\tau}_{X/S}$ is flat over S, or even only universally open over S, when X/S is simple. Here is an idea of an example, inspired by Igusa's surface. Let A/S be an abelian scheme, G a finite group of automorphisms of A. If G operates without fixed points on B/S projective and simple over S, with $\mathcal{O}_S \xrightarrow{\sim} g_*(\mathcal{O}_B)$, we construct $X = B \times_G \hat{A}$ which is an abelian scheme over $Y = B/G$, and one checks

$$\mathcal{P}ic_{X/S} \simeq \mathcal{P}ic_{Y/S} \times_S (\mathcal{P}ic_{\hat{A}/S})^G$$

(where upper G denotes the subscheme of invariants), hence

$$\boxed{\mathcal{P}ic^{\tau}_{X/S} \simeq \mathcal{P}ic^{\tau}_{Y/S} \times_S A^G}$$

Hence for getting examples of bad $\mathcal{P}ic^{\tau}_{X/S}$, we are led to study schemes of the type A^G, with S say spectrum of a discrete valuation ring V. Thus we are led to the questions:

a) Can it occur that there are components of $C = A^G$ which do not dominate S? For instance, $A_1^G =$ unit subgroup (set theoretically, or even scheme-theoretically) and $A_0^G \neq$ unit subgroup set theoretically—where A_0, A_1 are the special and the general fibers.
b) If $C_1 = A_1^G$ is connected (for instance is the unit subgroup), and hence $C^\circ = C_0^\circ \cup C_1^\circ$ is open, can it occur that C° is non flat over S [for instance $C_1 = \{e\}, C_0^\circ \neq \{e\}$]?
c) Same questions for $H^1(A, \mathcal{O}_{A/S})^G = t_{\hat{A}}{}^G$ and $H^0(A, \Omega^1_{A/S})^G = t_A{}^G$ (in order to get examples where the dimensions h^{01} and h^{10} for the fibers make a jump in the case of *equal characteristics*).

The trouble is I have no idea how to get non trivial ways of letting a finite group operate on an abelian variety. It seems that starting with products of elliptic curves and using only endomorphisms of the factors, for instance letting a finite subgroup of $\mathrm{GL}(n, R)$ operate on E^n, where R is the ring of endomorphisms of the elliptic curve E, won't give a counterexample (I more or less proved this latter statement). If p is the residue characteristic, one sees easily that the only trouble against flatness can come from a Sylow p-subgroup of G. For instance, in a) the question is equivalent to getting an example where $T_p(\overline{A_0}) \to T_p(\overline{A_1})$ (where T_p is the contravariant Tate functor, $T_p(M) = \mathrm{Hom}({}_{p^\infty}M, \mathbb{Q}_p/\mathbb{Z}_p)$, and $\overline{A_0}$ and $\overline{A_1}$ are the *geometric* fibers) induces

$$\hat{H}^{-1}(G, T_p(\overline{A_0})) \to \hat{H}^{-1}(G, T_p(\overline{A_1}))$$

which is *not injective*. I am convinced such things can happen. Perhaps you or Mumford are cleverer than I and find a counterexample? What I did get easily was an example of an *abelian* scheme X/S [product of two elliptic curves over S] such that multiplication $p : \mathcal{P}ic_{X/S} \to \mathcal{P}ic_{X/S}$ is *not* universally open, i.e. such that there exists an irreducible component C of $\mathcal{P}ic_{X/S}$ not dominating

S, but such that pC is contained in a component dominating S. [N.B. if n prime to all residue char., multiplication by n in any $\mathcal{P}ic_{X/S}$ is *étale*.]

Best regards to Karin, kids etc.

(signed) Schurik

P.S. I just proved: If $X \to S$ is *simple* and *projective*, then $\mathcal{P}ic^{\tau}_{X/S}$ is *projective* over S. Method:

a) From the fact that the fibers of $\mathcal{P}ic^0_{X/S}$ are proper, follows that $\mathcal{P}ic^0_{X/S}$ is proper over S, hence closed in $\mathcal{P}ic_{X/S}$, hence easily that $\mathcal{P}ic^{\tau}_{X/S}$ is *closed* in $\mathcal{P}ic_{X/S}$. It remains to prove it is of *finite type* over S—hence proper over S, and quasi-projective over S, hence projective.
b) For every $n > 0$, the kernel of $\mathcal{P}ic_{X/S} \xrightarrow{n} \mathcal{P}ic_{X/S}$ is of finite type over S [and even more: the multiplication μ by n is of finite type, hence finite]. If n is prime to the residue characteristics, this follows from the fact that μ is *étale* and has finite fibers. This reduces to the case S of char $p > 0$, $n = p$. Then I use a technique of descent involving the "relative p-power scheme" $(X/S)^{(p)}$, following a suggestion of Serre.
c) For variable $s \in S$ (S noetherian), the Néron-Severi torsion group of X_s remains of bounded order. This can be shown using the method of Matsusaka's proof for the finiteness of the "torsion group". From a), b), c), the theorem follows.

Remark: Using the Picard-Igusa inequality for $\rho =$ rank of Néron-Severi, and Lefschetz type theorems I told you about, one gets also that $\rho(X_s)$ remains bounded for $s \in S$ (S noetherian).
Question: Is $\mathcal{P}ic^{\tau}_{X/S}$ always of finite type over S, under merely the usual assumptions for existence of $\mathcal{P}ic_{X/S}$? I have no proof even if $X \to S$ is normal! Same question for ρ. This seems related to the question of uniform majorization of the Mordell-Weil-Néron-Lang finiteness theorem, for a *variable* abelian variety.