August 27, 1965 Alexandre Grothendieck

My dear Serre,

I am still turning algebraic cycles classes every which way in my head; from a technical point of view, I now see things more clearly, but the final breakthrough is still missing. As I know you are allergic to cohomologyJ.-P. Serre : "As I know you are allergic to cohomology…". Allergy, certainly not. Indigestion, perhaps?, I would like to show you the two key conjectures J.-P. Serre : This is a reference to the "standard conjectures"., which should be proved purely geometrically, i.e. without reference to cohomology. Let X denote a smooth connected projective n-dimensional variety over an algebraically closed field k, and Y a smooth hyperplane section. Let me denote by $C^i(X)$ the group of cycle classes of codimension i modulo algebraic equivalence, tensored with $\mathbf{Q}$, and by $\xi \in C^1(X)$ the class of Y.

Conjecture A: *For every integer i such that* $2i \leq n$, *the product with* ξ^{n-2i} *induces an isomorphism* $C^i(X) \cong C^{n-i}(X)$.

Modulo a verification which I have not written down in detail, to prove this conjecture it would be enough to prove surjectivity, and in fact to do it for $n = 2i + 1$. For the moment, the first case that eludes me is $i = 1$, $n = 3$. This conjecture would imply the analog, for the C^i, of the well-known Lefschetz theorems comparing the cohomologies of X and Y under the direct image and inverse image homomorphisms (to get the correct formulation all you need to remember is that C^i becomes H^{2i}), and it would simultaneously imply the cohomological Lefschetz theorems themselves, whose formally strongest form consists of asserting that ξ^{n-i} induces an isomorphism between H^i and H^{2n-i} for the cohomology with coefficients in $\mathbb{Q}_\ell$; let me point out that even this purely cohomological theorem is not yet proved! J.-P. Serre : "is not yet proved!". It was proved 15 years later by Deligne: *La conjecture de Weil* II, Publ. Math. IHES **52** (1980), 313–428, th. 4.1.1. The only thing that has been proved is the "weak" Lefschetz theorem, which says that the cohomological dimension of an affine variety of dimension n is bounded by n, which, for the affine variety $U = X - Y$ takes the form that $H^i(X) \to H^i(Y)$ is bijective for $i \leq n - 2$, and injective for $i = n - 1$, or alternatively $H^i(Y) \to H^{i+2}(X)$ is bijective for $i \geq n$ and surjective for $i = n - 1$. In fact, unless I am mistaken, I can deduce conjecture A from its following weaker form (which looks like a "weak" Lefschetz theorem):

CONJECTURE A$'$: *For every integer i such that $2i \geq n-1$, the direct image map $C^i(Y) \to C^{i+1}(X)$ is surjective, i.e. any algebraic cycle of dimension $j < n/2$ on X is τ-equivalent J.-P. Serre : "τ-equivalent" = equivalent up to torsion, i.e. after tensor product with* **Q**. *to the image of an algebraic cycle with rational coefficients on Y.*

Once again, unless I am mistaken, it is enough to work with $n = 2i + 1$. Moreover, in the argument used to reduce A to A$'$, it seems necessary to prove A$'$ over a not necessarily algebraically closed base field and for cycle classes which are rational over it, *or* restrict to stating A for cohomological equivalence alone (but A then loses the "purely geometric" nature I promised you!)

As I told you, A implies "the Künneth formula for cycles" (actually, I exaggerated a bit when I claimed that the two statements were equivalent). This then implies all the integrality theorems one could wish for (on coefficients of characteristic polynomials, for example), except that it seems possible to have powers of p= char. k in the denominator. This therefore implies the Weil conjectures, except for the question of the absolute values of eigenvalues, which will be covered by B. Note also that conjecture A appears to be the "minimum minimorum" to be able to give a usable rigorous definition of the concept of a motive over a field.

CONJECTURE B: *Assume that $n = 2m$, and let $P^m(X)$ be the kernel of the multiplication by ξ homomorphism from $C^m(X)$ to $C^{m+1}(X)$. Then the form $(-1)^m\epsilon(xx')$ on $P^m(X)$ is positive definite.*

For the Weil conjectures, it would actually be enough to show that this form is positive. But this stronger formulation also implies other attractive results, such as the fact that τ-equivalence = numerical equivalence (= homological equivalence with $\mathbb{Q}_\ell$ coefficients, since this one is sandwiched in between the other two), and the fact that the $C^m(X)$ are finite-dimensional, — and in fact, that the groups of τ-equivalence classes of cycles are finitely generated. (From this, one can formally deduce that that $C^m(X) \otimes \mathbb{Q}_\ell \to H^{2m}(X)(m)$ is injective, and thus the rank of $C^m(X)$ is bounded by $b_{2m}(X)$.) Furthermore, B also implies that the category of motives constructed from non-singular projective varieties is semi-simple, and in more down-to-earth terms, that the ring of classes of algebraic correspondences on X (modulo τ-equivalence and tensored with **Q**) is a finite semi-simple algebra over **Q** which can in fact be equipped with an involution and a trace satisfying the usual conditions. One also gets the following result, which may be viewed in some sense as a generalization of A (assuming that the cohomological version of A is already proved): Let $H^{2i}(X)(i) \to H^{2i+2r}(X')(i+r)$ be a homomorphism defined by an algebraic correspondence class, and let $C^i(X) \to C^{i+r}(X')$ be the homomorphism it defines on cycles. Then an element of $C^{i+r}(X')$ lies in the image of $C^i(X)$ if and only if its image in H^{2i+2r} lies in that of H^{2i}. Note also that the semi-simplicity result mentioned above would imply the analogous semi-simplicity result for the Galois group actions in the Tate conjectures. Finally, I have more or less convinced myself that A and B also imply a reasonable theory of the abelian varieties which appear as parameter varieties of continuous families of algebraic cycles, and in particular allow us to obtain the necessary relations between these "intermediate Jacobians" (which can be viewed as "algebraic pieces" in Weil's Jacobians) and the cohomology in odd degree. It is however quite possible that the proof of B will itself be linked to the introduction of these abelian varieties.

In any case, A and A′ seem to me to be in some sense preliminary to B, and it seems reasonable to start with them. Let me point out a rather suggestive statement which is equivalent to A′:

Conjecture A″: *Let U be the n-dimensional affine open set $X - Y$, then $C^i(U) = 0$ for every i such that $2i > n$.*

In fact, it is possible that this holds for any smooth affine variety, and even that for any smooth variety (affine or not), every cycle which is cohomologically equivalent to 0 is τ-equivalent to 0 (as I said, this would follow for projective varieties from theorems A and B).

For the moment, what is needed is to invent a process for deforming a cycle whose dimension is not too large, in order to push it to infinity. Perhaps you would like to think about this yourself? I have only just started on it today, and am writing to you because I have no ideas.

Yours,

A. Grothendieck