August 18, 1959 Alexandre Grothendieck

My dear Serre, Tate has written to me about his elliptic curves stuff, and has asked me if I had any ideas for a global definition of analytic varieties J.-P. Serre : This is a reference to the p-adic theory of "Tate curves", which is at the origin of rigid analytic geometry.

Tate's text, which was written in 1959, was published (and completed) in 1995: J. Tate, *A review of non-archimedean elliptic functions*, in "Elliptic Curves, Modular Forms and Fermat's Last Theorem" (J. Coates and S. T. Yau, eds.), Intern. Press, Boston, 1995, 162–184. over complete valuation fields. I must admit that I have absolutely not understood why his results might suggest the existence of such a definition, and I remain skeptical. Nor do I have the impression of having understood his theorem at all; it does nothing more than exhibit, via brute formulas, a certain isomorphism of analytic groups; one could conceive that other equally explicit formulas might give another one which would be no worse than his (until proof to the contrary!)

I have given just enough thought to the "infinitesimal" part of the fundamental group to convince myself that it exists and is reasonable. Here is a (surely insufficient, actually) context in which it works. Take a connected scheme S (for instance an algebraic scheme over a field), an auxiliary category $\mathcal{C}$ (for instance the category of finite algebraic schemes over k), a category $\mathcal{G}$ whose objects are $\mathcal{C}$-groups, and whose morphisms are $\mathcal{C}$-group morphisms (for instance finite algebraic groups over k). Assume that fiber products exist in $\mathcal{C}$ and that $\mathcal{G}$ satisfies the following properties: (i) $\mathcal{G}$ is stable under products, and if $u, v : G \to G'$ are morphisms in $\mathcal{G}$, then the kernel of the pair (u, v), i.e. the maximal subgroup on which they coincide — which exists, since it can be expressed using a fiber product — lies in $\mathcal{G}$. (ii) If $u : G \to G'$ is a morphism in $\mathcal{G}$, then the image group exists, is isomorphic to a quotient of G (as it should be), and belongs to $\mathcal{G}$. (iii) Every decreasing sequence of subgroups $\in \mathcal{G}$ of a $G \in \mathcal{G}$ is eventually stationary. Moreover, assume given a covariant functor F from $\mathcal{G}$ to group schemes over S (for instance $G \to S \times_k G$), and assume that : (iv) The functor F commutes with products, kernels of pairs of morphisms and images (one could say that F is "exact"). (v) If H is of the form $F(G)$ $(G \in \mathcal{G})$, then H is "special", i.e. there is an exact sequence of finite flat group schemes over S,

$$e \to H_{\text{inf}} \to H \to H_{\text{sep}} \to 0$$

where H_{inf} is purely infinitesimal (i.e. the projection $H_{\text{inf}} \to S$ is geometrically injective) and H_{sep} is unramified over S. (In the case of a base field k, such an exact sequence can be deduced from an analogous exact sequence for a finite algebraic group G over k; moreover, if k is perfect, this sequence splits canonically, since G_{red} is then a subgroup of G which is isomorphic to G_{sep} under the projection $G \to G_{\mathrm{sep}}$. Note, however, that G_{red} does act on G_{inf}; one has only a semi-direct product). Finally, (vi) S is reduced.

Conditions (v) and (vi) look ugly, and are essentially temporary. They are useful because of this

LEMMA: *Let H be as in* (v)*, S as in* (vi)*, P a homogeneous principal H-bundle, Q another, u and v two isomorphisms from P to Q taking some given "marked point" into another one; then $u = v$.*

By twisting H, the problem can be reduced to the case where P is trivial, and the result then follows from the following fact: a section of Q is an isomorphism between S and a connected component of Q_{red}.

Note, however, that conditions (i)-(vi) do not preclude the possibility of twisted structural groups (with respect to a base field k).

Now let a be a "marked point" of S (i.e. an algebraically closed extension of the residue field of some $s \in S$). For every $G \in \mathcal{G}$, let $Z(S, a; G)$ or simply $Z(G)$ denote the set of classes (up to isomorphism) of homogeneous principal bundles over S with group $F(G)$, equipped with a marked point over a. Obviously $Z(G)$ is a functor from $\mathcal{G}$ into the category of sets. This functor has the following properties: 1) It commutes with products (since F does); 2) It commutes with kernels of pairs: in other words, if $G'' \longrightarrow G \overset{u}{\underset{v}{\rightrightarrows}} G'$ is exact in $\mathcal{G}$, then $Z(G'') \longrightarrow Z(G) \overset{Z(u)}{\underset{Z(v)}{\rightrightarrows}} Z(G')$ is exact, i.e. $Z(G'')$ can be identified with the set of elements in $Z(G)$ whose images in $Z(G')$ under $Z(u)$ and $Z(v)$ are the same. (This follows from the exactness of F and the *lemma*).

It follows formally from these two properties that one can find a filtered projective system $(G_i)_{i\in I}$ in $\mathcal{G}$, with morphisms $G_i \to G_j$ which are epimorphisms in $\mathcal{C}$, and which is "essentially unique" in an easily specified sense, such that there is a functorial isomorphism

$$Z(G) = \varinjlim \mathrm{Hom}_{\mathcal{G}}(G_i, G).$$

It is this projective system (considered modulo an equivalence which intuitively means that one passes to the projective limit of the G_i) which may be denoted $\pi_1^{\mathcal{G}}(S, a)$ and which plays the role of the fundamental group of S at a (with respect to the category $\mathcal{G}$ of groups, equipped with the functor F). In the base field k case, where $\mathcal{G}$ is the category of finite algebraic groups over k, one could write[3] $\pi_1(S/k, a)$: it is the proalgebraic fundamental group (with infinitesimal part) of the k-scheme S. When k is perfect, the decomposition of a finite algebraic group into its infinitesimal part and its reduced part shows that the fundamental group is the semi-direct product of its reduced or separable part (which corresponds to an ordinary discontinuous compact group on which the ordinary fundamental group of k — i.e. the Galois group of $\overline{k}$ over k — acts) with its infinitesimal part, which actually no longer depends on the choice of the base point a of S. Note that the separable part of the fundamental group can easily be recovered using the ordinary fundamental group of S and its natural homomorphism into $\mathrm{Gal}(\overline{k}/k)$, but is "larger" than the ordinary fundamental group, since it corresponds to the classification of principal coverings with structural group not only an ordinary finite group, but also a finite group which is separable over k (i.e. an ordinary finite group on which $\mathrm{Gal}(\overline{k}/k)$ acts not necessarily trivially.) I admit that if k is not algebraically closed, then the fundamental group described above is probably not the right one; one should probably consider the fundamental group of $S \otimes_k \overline{k}$, which is a proalgebraic group defined over $\overline{k}$; note that it is equipped with descent data from $\overline{k}$ to k and thus is actually defined over k. This would work whenever the base point is *k-rational*. In any case, there should be a "pro-group scheme" $\underline{\pi}_1(S)$ over S (a local system of fundamental groups). 3/4

Here is how to prove that the projective system (G_i) exists. A pair (G, z) with $G \in \mathcal{G}$ and $z \in Z(G)$ is said to be *minimal* if there is no subgroup $G' \subset G$, $G' \neq G$, such that z is of the form $Z(i)(z')$ where $z' \in Z(G')$ and $i : G' \to G$ is the inclusion. Let us say that a pair (G, z) is *dominated* by a pair (G', z') if there is a homomorphism $u : G' \to G$ such that $z = Z(u)(z')$. It follows from (iii) that every pair (G, z) is dominated by a minimal pair, and from property 2) of Z that if (G', z') is minimal and dominates (G, z), then there is a *unique* $u : G' \to G$ such that $z = Z(u)(z')$. From this it follows that the minimal pairs (G, z) form a projective system for the relation of domination, which is in fact a directed set (since (G, z) and (G', z') are dominated by $(G \times G', (z, z'))$, which is itself dominated by some minimal (G'', z'')): this is the desired system. If one wants, one can choose a pair (G, z) in every system of isomorphic minimal pairs, in such a way that the set I of indices becomes ordered and not simply pre-ordered. (N.B. This is also the kind of formal argument that is used in "moduli theory"...).

[3] note in the margin: wrongly!

I do not yet know how to formulate the homotopy exact sequence; to do it right, the inclusion of an infinitesimal part into the fundamental group should provide a satisfactory theory of the behavior of the fundamental group under specialization. I hope that if $f : X \to Y$ is a proper separable morphism with absolutely connected fibers, i.e. such that $f(\mathcal{O}_X) = \mathcal{O}_Y$, where for simplicity's sake X is equipped with a section s over Y which determines base points on the fibers, then the total fundamental groups of the fibers of X form a pro-group scheme over Y $\pi_1(X/Y, S)$: more precisely, I hope that it is possible to find a filtered projective system $(G_i)_{i\in I}$ of "special" group schemes G_i over Y, and homomorphisms $G_i \to G_j$ which are epimorphisms of Y-schemes (i.e. correspond to an injective homomorphism of coherent sheaves on Y), such that the total fundamental groups of the fibers of X can be deduced from the said pro-group scheme simply by specializing. (In any case, this is what the case where X is an abelian scheme over Y seems to suggest, cf. below.) Even without choosing a section, one could deduce the existence *over* X of a pro-group scheme $\underline{\pi}_1(X/Y)$, "the fundamental group of the fibers". When there is no base field k, however, it is not clear how and to which fundamental groups of X and Y one can attach the fundamental groups of the fibers to replace an exact homotopy sequence. If there is a base field, an initial conjecture would be that $\underline{\pi}_1(X/Y)$, the pro-group scheme $\pi_1(X)$ over X of local systems of fundamental groups of X at its various points, and the preimage under f of the analogous pro-group scheme $\underline{\pi}_1(Y)$ on Y, are related by an exact sequence. Have fun with higher homotopy groups...

It is not said that the conjecture I look for is more difficult to prove than the part that is already known for ordinary fundamental groups. It would imply that for complete schemes X and Y over an algebraically closed field k, $\pi_1(X \times Y/k) = \pi_1(X/k) \times \pi_1(Y/k)$. Using your arguments, it follows that if X is an abelian variety over k, then $\pi_1(X/k)$ is abelian, and a *minimal* principal covering X' of X is an abelian variety which is isogenous to X. It follows that

$$\pi_1(X/k) = \varprojlim {}_nX,$$

where ${}_nX$ is the kernel (with its infinitesimal part as well, of course) of multiplication by n, the homomorphism of ${}_{mn}X$ into ${}_nX$ being induced by multiplication by m. (By Cartier's results, this fundamental group is dual, in Cartier's sense of the word, to the ind-algebraic group which is the inductive-limit of the ${}_nX^*$, where X^* is the dual variety of X.) Taking the p-component of this fundamental group, one obtains something which, for a prime number p, should play the role of the Weil module. There is no doubt that it is possible (and Cartier must have already done it) to functorially associate J.-P. Serre : "It is possible ... a module over the Witt ring". This was done (and continues to be done in increasing generality) by Cartier, Gabriel, Manin, Fontaine ... See for instance Chap.V of the work by Demazure and Gabriel quoted in note 9.2. a module over the Witt ring to an abelian infinitesimal pro-algebraic group, and that it is easy to check that the module decomposes into the three parts you know (corresponding to the three principal types of abelian algebraic p-groups). Thus, one obtains a more natural formulation of your theorem (which should give a universal proof which does not distinguish the cases $\ell \neq p$ and $\ell = p$), and at the same time, one sees that your bewildering sum behaves well when X varies in a family, i.e. if X is an abelian scheme over Y (since in this case the ${}_nX$ are wonderful finite flat group schemes over Y, whose projective limit can be taken formally...).

Next year, I hope to getJ.-P. Serre : "Next year, I hope to get..." Hope springs eternal! This reminds one of the beginnings of Bourbaki, who had hoped to be done within a few years.

In fact, the EGA's stopped after chapter IV, a text of almost 800 pages, whose last part came out in 1967. a satisfactory theory of the fundamental group, and finish up the writing of chapters IV, V, VI, VII (the last one being the fundamental group) at the same time as categories. In two years, residues, duality, intersections, Chern, and Riemann-Roch. In three years, Weil cohomology, and a little homotopy, God willing. In between, I don't know when, the "big existence theorem" with Picard etc., some algebraic curves, and abelian schemes. Unless there are unexpected difficulties or I get bogged down, the multiplodocus should be ready in 3 years time, or 4 at the outside. We will then be able to start doing algebraic geometry! Yours, A. Grothendieck