Les Aumettes 19.2.1983

Dear Daniel,

**1** Last year Ronnie Brown from Bangor sent me a heap of reprints and preprints by him and a group of friends, on various foundational matters of homotopical algebra. I did not really dig through any of this, as I kind of lost contact with the technicalities of this kind (I was never too familiar with the homotopy techniques anyhow, I confess) – but this reminded me of a few letters I had exchanged with Larry Breen in 1975, where I had developed an outline of a program for a kind of "topological algebra", viewed as a synthesis of homotopical and homological algebra, with special emphasis on topoi – most of the basic intuitions in this program arising from various backgrounds in algebraic geometry. Some of those intuitions we discussed, I believe, at IHES eight or nine years before, at a time when you had just written up your nice ideas on axiomatic homotopical algebra, published since in Springer's Lecture Notes. I write you under the assumption that you have not entirely lost interest for those foundational questions you were looking at more than fifteen years ago. One thing which strikes me, is that (as far as I know) there has not been any substantial progress since – it looks to me that an understanding of the basic structures underlying homotopy theory, or even homological algebra only, is still lacking – probably because the few people who have a wide enough background and perspective enabling them to feel the main questions, are devoting their energies to things which seem more directly rewarding. Maybe even a wind of disrepute for any foundational matters whatever is blowing nowadays! In this respect, what seems to me even more striking than the lack of proper foundations for homological and homotopical algebra, is the absence I daresay of proper foundations for topology itself! I am thinking here mainly of the development of a context of "tame" topology, which (I am convinced) would have on the everyday technique of geometric topology (I use this expression in contrast to the topology of use for analysts) a comparable

*The importance of innocence.*

impact or even a greater one, than the introduction of the point of view of schemes had on algebraic geometry. The psychological drawback here I believe is not anything like messiness, as for homological and homotopical algebra (as for schemes), but merely the inrooted inertia which prevents us so stubbornly from looking innocently, with fresh eyes, upon things, without being dulled and imprisoned by standing habits of thought, going with a familiar context – *too* familiar a context! The task of working out the foundations of tame topology, and a corresponding structure theory for "stratified (tame) spaces", seems to me a lot more urgent and exciting still than any program of homological, homotopical or topological algebra.

**2** The motivation for this letter was the latter topic however. Ronnie Brown and his friends are competent algebraists and apparently strongly motivated for investing energy in foundational work, on the other hand they visibly are lacking the necessary scope of vision which geometry alone provides. They seem to me kind of isolated, partly due I guess to the disrepute I mentioned before – I suggested to try and have contact with people such as yourself, Larry Breen, Illusie and others, who have the geometric insight and who moreover, may not think themselves too good for indulging in occasional reflection on foundational matters and in the process help others do the work which should be done.

*A short look into purgatory.*

At first sight it has seemed to me that the Bangor group had indeed come to work out (quite independently) one basic intuition of the program I had envisioned in those letters to Larry Breen – namely that the study of $n$-truncated homotopy types (of semisimplicial sets, or of topological spaces) was essentially equivalent to the study of so-called $n$-groupoids (where $n$ is any natural integer). This is expected to be achieved by associating to any space (say) $X$ its "fundamental $n$-groupoid" $\Pi_n(X)$, generalizing the familiar Poincaré fundamental groupoid for $n = 1$. The obvious idea is that 0-objects of $\Pi_n(X)$ should be the points of $X$, 1-objects should be "homotopies" or paths between points, 2-objects should be homotopies between 1-objects, etc. This $\Pi_n(X)$ should embody the $n$-truncated homotopy type of $X$, in much the same way as for $n = 1$ the usual fundamental groupoid embodies the 1-truncated homotopy type. For two spaces $X, Y$, the set of homotopy-classes of maps $X \to Y$ (more correctly, for general $X, Y$, the maps of $X$ into $Y$ in the homotopy category) should correspond to $n$-equivalence classes of $n$-functors from $\Pi_n(X)$ to $\Pi_n(Y)$ – etc. There are very strong suggestions for a nice formalism including a notion of geometric realization of an $n$-groupoid, which should imply that any $n$-groupoid (or more generally of an $n$-category) is relativized over an arbitrary topos to the notion of an $n$-gerbe (or more generally, an $n$-stack), these become the natural "coefficients" for a formalism of non-commutative cohomological algebra, in the spirit of Giraud's thesis.

But all this kind of thing for the time being is pure heuristics – I never so far sat down to try to make explicit at least a definition of $n$-categories and $n$-groupoids, of $n$-functors between these etc. When I got the Bangor reprints I at once had the feeling that this kind of work

had been done and the homotopy category expressed in terms of $\infty$-groupoids. But finally it appears this is not so, they have been working throughout with a notion of $\infty$-groupoid too restrictive for the purposes I had in mind (probably because they insist I guess on strict associativity of compositions, rather than associativity up to a (given) isomorphism, or rather, homotopy) – to the effect that the simply connected homotopy types they obtain are merely products of Eilenberg-MacLane spaces, too bad! They do not seem to have realized yet that this makes their set-up wholly inadequate to a sweeping foundational set-up for homotopy. This brings to the fore again to work out the suitable definitions for $n$-groupoids – if this is not done yet anywhere. I spent the afternoon today trying to figure out a reasonable definition, to get a feeling at least of where the difficulties are, if any. I am guided mainly of course by the topological interpretation. It will be short enough to say how far I got. The main part of the structure it seems is expressed by the sets $F_i$ ($i \in \mathbb{N}$) of $i$-objects, the source, target and identity maps

$$\begin{aligned} \mathrm{s}_1^i, \mathrm{t}_1^i &: F_i \to F_{i-1} \quad (i \geq 1) \\ \mathrm{k}_1^i &: F_i \to F_{i+1} \quad (i \in \mathbb{N}) \end{aligned}$$

and the symmetry map (passage to the inverse)

$$\mathrm{inv}_i : F_i \to F_i \quad (i \geq 1),$$

satisfying some obvious relations: $\mathrm{k}_1^i$ is right inverse to the source and target maps $\mathrm{s}_1^{i+1}, \mathrm{t}_1^{i+1}$, $\mathrm{inv}_i$ is an involution and "exchanges" source and target, and moreover for $i \geq 2$

*notation:*
$d_a = \mathrm{k}_1^i(a)$,
$\check{u} = \mathrm{inv}_i(u)$

$$\begin{aligned} \mathrm{s}_1^{i-1}\mathrm{s}_1^i &= \mathrm{s}_1^{i-1}\mathrm{t}_1^i \left(\overset{\mathrm{def}}{=} \mathrm{s}_2^i : F_i \to F_{i-2}\right) \\ \mathrm{t}_1^{i-1}\mathrm{s}_1^i &= \mathrm{t}_1^{i-1}\mathrm{t}_1^i \left(\overset{\mathrm{def}}{=} \mathrm{t}_2^i : F_i \to F_{i-2}\right); \end{aligned}$$

thus the composition of the source and target maps yields, for $0 \leq j \leq i$, just *two* maps

$$\mathrm{s}_\ell^i, \mathrm{t}_\ell^i : F_i \to F_{i-\ell} = F_j \quad (\ell = i - j).$$

The next basic structure is the composition structure, where the usual compoisition of arrows, more specifically of $i$-objects ($i \geq 1$) $v \circ u$ (defined when $\mathrm{t}_1(u) = \mathrm{s}_1(v)$) must be supplemented by the Godement-type operations $\mu * \lambda$ when $\mu$ and $\lambda$ are "arrows between arrows", etc. Following this line of thought, one gets the composition maps

$$(u, v) \mapsto (v *_\ell u) : (F_i, \mathrm{s}_\ell^i) \times_{F_{i-\ell}} (F_i, \mathrm{s}_\ell^i) \to F_i,$$

the composition of $i$-objects for $1 \leq \ell \leq i$, being defined when the $\ell$-target of $u$ is equal to the $\ell$-source of $v$, and then we have

$$\left.\begin{aligned} \mathrm{s}_1^i(v *_\ell u) &= \mathrm{s}_1^i(v) *_{\ell-1} \mathrm{s}_1^i(u) \\ \mathrm{t}_1^i(v *_\ell u) &= \mathrm{t}_1^i(v) *_{\ell-1} \mathrm{s}_1^i(u) \end{aligned}\right\} \quad \ell \geq 2 \text{ i.e. } \ell - 1 \geq 1$$

and for $\ell = 1$

$$\mathrm{s}_1(v *_1 u) = s_1(u)$$
$$\mathrm{t}_1(v *_1 u) = t_1(v)$$

(NB the operation $v *_1 u$ is just the usual composition $v \circ u$).

One may be tempted to think that the preceding data exhaust the structure of $\infty$-groupoids, and that they will have to be supplemented only by a handful of suitable *axioms*, one being *associativity* for the operation $\overset{i}{\underset{\ell}{*}}$, which can be expressed essentially by saying that that composition operation turns $F_i$ into the set of arrows of a category having $F_{i-\ell}$ as a set of objects (with the source and target maps $\mathrm{s}^i_\ell$ and $\mathrm{t}^i_\ell$, and with identity map $\mathrm{k}^{i-\ell}_\ell : F_{i-\ell} \to F_i$ the composition of the identity maps $F_{i-\ell} \to F_{i-\ell+1} \to \cdots \to F_{i-1} \to F_i$), and another being the Godement relation

$$(v' *_\alpha v) *_\nu (u' *_\alpha u) = (v' *_\nu u') *_\alpha (v *_\nu u)$$

(with the assumptions $1 \leq \alpha \leq \nu$, and $u, u', v, v'' in F_i$ and

$$\begin{cases} \mathrm{t}_\alpha(u) = \mathrm{s}_\alpha(u') \\ \mathrm{t}_\alpha(v) = \mathrm{s}_\alpha(v') \end{cases} \quad \mathrm{t}_\nu(u) = \mathrm{s}_\nu(v) = \mathrm{s}_\nu(v') = \mathrm{t}_\nu(u')$$

implying that both members are defined), plus the two relations concerning the inversion of $i$-objects ($i \geq 1$) $u \mapsto \check{u}$,

$$u *_1 \check{u} \;=\; \mathrm{id}_{\mathrm{t}_1(u)}, \qquad \check{u} *_1 u \;=\; \mathrm{id}_{\mathrm{s}_1(u)}, \qquad (\check{v} *_\ell \check{u}) \;=\; ? \quad (\ell \;\geq\; 2)$$

It just occurs to me, by the way, that the previous description of basic (or "primary") data for an $\infty$-groupoid is already incomplete in some rather obvious respect, namely that the symmetry-operation $\mathrm{inv}_i : u \mapsto \check{u}$ on $F_i$ must be complemented by $i-1$ similar involutions on $F_i$, which corresponds algebraically to the intuition that when we have an $(i+1)$-arrow $\lambda$ say between two $i$-arrows $u$ and $v$, then we must be able to deduce from it another arrow from $\check{u}$ to $\check{v}$ (namely $u \mapsto \check{u}$ has a "functorial character" for variable $u$)? This seems a rather anodine modification of the previous set-up, and is irrelevant for the main point I want to make here, namely: that for the notion of $\infty$-groupoids we are after, all the equalities just envisioned in this paragraph (and those I guess which will ensure naturality by the necessary extension of the basic involution on $F_i$) should be replaced by "homotopies", namely by $(i+1)$-arrows between the two members. These arrows should be viewed, I believe, as being part of the data, they appear here as a kind of "secondary" structure. The difficulty which appears now is to work out the natural coherence properties concerning this secondary structure. The first thing I could think of is the "pentagon axiom" for the associativity data, which occurs when looking at associativities for the compositum (for $\overset{i}{\underset{\ell}{*}}$ say) of four factors. Here again the first reflex would be to write down, as usual, an *equality* for two compositions of associativity isomorphisms, exhibited in the pentagon diagram. One suspects however that such

equality should, again, be replaced by a "homotopy"-arrow, which now appears as a kind of "ternary" structure – before even having exhausted the list of coherence "relations" one could think of with the respect to the secondary structure! Here one seems caught at first sight in an infinite chain of ever "higher", and presumably, messier structures, where one is going to get hopelessly lost, unless one discovers some simple guiding principle for shedding some clarity in the mess.

**3** I thought of writing you mainly because I believe that, if anybody, you should know if the kind of structure I am looking for has been worked out – maybe even *you* did? In this respect, I vaguely remember that you had a description of $n$-categories in terms of $n$-semisimplicial sets, satisfying certain exactness conditions, in much the same way as an ordinary category can be interpreted, via its "nerve", as a particular type of semisimplicial set. But I have no idea if your definition applied only for describing $n$-categories with strict associativities, or not.

*"Fundamental $\infty$-groupoids" as objects of a "model category"?*

Still some contents in the spirit of your axiomatics of homotopical algebra – in order to make the question I am proposing more seducing maybe to you! One comment is that presumably, the category of $\infty$-groupoids (which is still to be defined) is a "model category" for the usual homotopy category; this would be at any rate one plausible way to make explicit the intuition referred to before, that a homotopy type is "essentially the same" as an $\infty$-groupoid up to $\infty$-equivalence. The other comment: the construction of the fundamental $\infty$-groupoid of a space, disregarding for the time being the question of working out in full the pertinent structure on this messy object, can be paraphrased in any model category in your sense, and yields a functor from this category to the category of $\infty$-groupoids, and hence (by geometric realization, or by localization) also to the usual homotopy category. Was this functor obvious beforehand? It is of a non-trivial nature only when the model category is *not* pointed – as a matter of fact the whole construction can be carried out canonically, in terms of a "cylinder object" $I$ for the final object $e$ of the model category, playing the role of the unit argument. It's high time to stop this letter – please excuse me if it should come ten or fifteen years too late, or maybe one year too early. If you are not interested for the time being in such general nonsense, maybe you know someone who is ...

Very cordially yours

20.2.

I finally went on pondering about a definition of $\infty$-groupoids, and it seems to me that, after all, the topological motivation does furnish the "simple guiding principle" which yesterday seemed to me to be still to be discovered, in order not to get lost in the messiness of ever higher order structures. Let me try to put it down roughly.

**4** First I would like to correct somewhat the rather indiscriminate description I gave yesterday of what I thought of viewing as "primary", secondary, ternary etc. structures for an $\infty$-groupoid. More careful reflection conduces to view as the most primitive, starting structure on the set of sets $F_i$ $(i \in \mathbb{N})$, as a skeleton on which progressively organs and flesh will be added, the mere data of the source and target maps

*A bit of ordering in the mess of "higher order structures".*

$$\mathrm{s}_i, \mathrm{t}_i : F_i \rightrightarrows F_{i-1} \quad (i \geq 1),$$

which it will be convenient to supplement formally by corresponding maps $\mathrm{s}_0, \mathrm{t}_0$ for $i = 1$, from $F_0$ to $F_{-1} \overset{\text{def}}{=}$ one-element set. In a moment we will pass to a universal situation, when the $F_i$ are replaced by the corresponding "universal" objects $\boldsymbol{F}_i$ in a suitable category stable under finite products, where $F_{-1}$ will be the final element. For several reasons, it is not proper to view the inversion maps $\mathrm{inv}_i : F_i \to F_i$, and still less the other $i-1$ involutions on $F_i$ which I at first overlooked, as being part of the primitive or "skeletal" structure. One main reason is that already for the most usual 2-groupoids, such as the 2-groupoid whose 0-objects are ordinary (1-)groupoids, the 1-objects being equivalences between these (namely functors which are fully faithful and essentially surjective), and the 2-objects morphisms (or "natural transformations") between such, there is *not*, for an $i$-object $f : C \to C'$, a natural choice of an "inverse" namely of a quasi-inverse in the usual sense. And even assuming that such quasi-inverse is chosen for every $f$, it is by no means clear that such choice can be made involutive, namely such that $(f^\vee)^\vee = f$ for every $f$ (and not merely $(f^\vee)^\vee$ isomorphic to $f$). The maps $\mathrm{inv}_i$ will appear rather, quite naturally, as "primary structure", and they will not be involutions, but "pseudo-involutions" (namely involutions "up to homotopy"). It turns out that among the various functors that we will construct, from the category of topological spaces to the category of $\infty$-groupoids (the construction depending on arbitrary choices and yielding a large bunch of mutually non-isomorphic functors, which however are "equivalent" in a sense which will have to be made precise) – there are choices neater than others, and some of these will yield in the primary structure maps $\mathrm{inv}_i$ which are actual involutions and similarly for the other pseudo-involutions, appearing in succession as higher order structure. The possibility of such neat and fairly natural choices had somewhat misled me yesterday.

What may look less convincing though at first sight, is my choice to view as non-primitive even the "degeneration maps" $\mathrm{k}_1^i : F_i \to F_{i+1}$, associating to every $i$-objects the $i+1$-object acting as an identity on the former. In all cases I have met so far, these maps are either given beforehand with the structure (of a 1-category or 2-category say), or they can be uniquely deduced from the axioms. In the present set-up however, they seem to me to appear more naturally as "primary" (not as primitive) structure, much in the same way as the $\mathrm{inv}_i$. Different choices for associating an $\infty$-groupoid to a topological space, while yielding the same base-sets $F_i$, will however (according to this point of view) give rise to different maps $\mathrm{k}_1^i$. The main motivation for this point of view

comes from the fact that the mechanism for a uniform construction of the chain of ever higher order structures makes a basic use of the source and target maps only and of the "transposes" (see below), and (it seems to me) not at all of the degeneration maps, which in this respect rather are confusing the real picture, if viewed as "primitive". The degeneration maps rather appear as typical cases of primary structure, probably of special significance in the practical handling of $\infty$-groupoids, but not at all in the conceptual machinery leading up to the construction of the structure species of "$\infty$-groupoids".

Much in the same way, the composition operations $\overset{i}{\underset{\ell}{*}}$ are viewed as primary, not as primitive or skeletal structure. Their description for the fundamental $\infty$-groupoid of a space – for instance the description of composition of paths – depends on arbitrary choices, such as the choice of an isomorphism (say) between $(I,1) \amalg_e (I,0)$ and $I$, where $I$ is the unit interval, much in the same way as the notion of an inverse of a path depends on the choice of an isomorphism of $I$ with itself, exchanging the two end-points 0 and 1. The operations $\overset{i}{\underset{2}{*}}$ of Godement take sense only once the composition operations $*_1$ are defined – they are "secondary structure", and successively the operations $*_3, \ldots, *_i$ appear as ternary etc. structure. This is correctly suggested by the notations which I chose yesterday, where however I hastily threw all the operations into a same pot baptized "primary structure"!

**5** It is about time though to come to a tentative precise definition of description of the process of stepwise introduction of an increasing chain of higher order structure. This will be done by introducing a canonical sequence of categories and functors *Jumping over the abyss!*

$$C_0 \to C_1 \to C_2 \to \cdots \to C_n \to C_{n+1} \to \cdots,$$

where $C_n$ denotes the category harbouring the "universal" partial structure of a would-be $\infty$-groupoid, endowed only with its "structure of order $\leq n$". The idea is to give a direct inductive construction of this sequence, by describing $C_0$, and the passage from $C_n$ to $C_{n+1}$ $(n \geq 0)$, namely from an $n$-ary to $(n+1)$-ary structure. As for the meaning of "universal structure", once a given structure species is at hand, it depends on the type of categories (described by the exactness properties one is assuming for these) one wants to take as carriers for the considered structure, and the type of exactness properties one assumes for the functor one allows between these. The choice depends partly on the particular species; if it is an algebraic structure which can be described say by a handful of composition laws between a bunch of base sets (or base-objects, when looking at "realizations" of the structure not only in the category (Sets)), one natural choice is to take categories with finite products, and functors which commute to these. For more sophisticated algebraic structures (including the structure of category, groupoid or the like), which requires for the description of data or axioms not only finite products, but also some fiber products, one other

familiar choice is to take categories with finite inverse limits, and left exact functors. Still more sophisticated structures, when the description of the structure in terms of base objects requires not only some kind of inverse limits, but also more or less arbitrary direct limits (such as the structure of a comodule over an algebra, which requires consideration of tensor products over a ring object), still more stringent conditions will have to be imposed upon categories and functors between these, for the structure to make a sense in these categories, and the functors to transform a structure of this type in one category into one of same type in another. In most examples I have looked up, everything is OK taking categories which are topoi, and functors between these which are inverse image functors for morphisms of topoi, namely which are left exact and commute with arbitrary direct limits. There is a general theorem for the existence of universal structures, covering all these cases – for instance there is a "classifying topos" for most algebro-geometric structures, whose cohomology say should be viewed as the "classifying cohomology" of the structure species considered. In the case we are interested in here, it is convenient however to work with the smallest categories $C_n$ feasible – which amounts to being as generous as possible for the categories one is allowing as carriers for the structure of an $\infty$-groupoid, and for the functors between these which are expected to carry an $\infty$-groupoid into an $\infty$-groupoid. What we will do is define ultimately a structure of an $\infty$-groupoid in a category $C$, as a sequence of objects $F_i$ ($i \in \mathbb{N}$), endowed with some structure to be defined, assuming merely that in $C$ finite products of the $F_i$ exist, plus certain finite inverse limits built up with the $F_i$'s and the maps $\mathrm{s}^i_\ell$, $\mathrm{t}^i_\ell$ between them (the iterated source and target maps). It should be noted that the type of $\varprojlim$ we allow, which will have to be made precise below, is fixed beforehand in terms of the "skeletal" or "primitive" structure alone, embodied by the family of couples $(\mathrm{s}_i, \mathrm{t}_i)_{i\in\mathbb{N}}$. This implies that the categories $\mathbb{C}_i$ can be viewed as having *the same set of objects*, namely the objects $\boldsymbol{F}_i$ (written in bold now to indicate their universal nature, and including as was said before $\boldsymbol{F}_{-1}$ = final object), plus the finite products and iterated fiber products of so-called "standard" type. While I am writing, it appears to me even that the finite products here are of no use (so we just drop them both in the condition on categories which are accepted for harbouring $\infty$-groupoids, and in the set of objects of the categories $C_i$). Finally, the common set of objects of the categories $\mathbb{C}_i$ is the set of "standard" iterated fiber products of the $\boldsymbol{F}_i$, built up using only the primitive structure embodied by the maps $\mathrm{s}_i$ and $\mathrm{t}_i$ (which I renounce to underline!). This at the same time gives, in principle, a precise definition of $\mathbb{C}_0$, at least up to equivalence – it should not be hard anyhow to give a wholly explicit description of $\mathbb{C}_0$ as a small category, having a countable set of objects, once the basic notion of the standard iterated fiber-products has been explained.

or torsors under a group...

Once $\mathbb{C}_0$ is constructed, we will get the higher order categories $\mathbb{C}_1$ (primary), $\mathbb{C}_2$ etc. by an inductive process of *successively adding arrows*. The category $\mathbb{C}_\infty$ will then be defined as the direct limit of the categories

$\mathbb{C}_n$, having the same objects therefore as $\mathbb{C}_0$, with

$$\mathrm{Hom}_\infty(X,Y) = \varinjlim_n \mathrm{Hom}_n(X,Y)$$

for any two objects. This being done, giving a structure of $\infty$-groupoid in any category $C$, will amount to giving a functor

$$\mathbb{C}_\infty \to C$$

commuting with the standard iterated fiber-products. This can be reexpressed, as amounting to the same as to give a sequence of objects $(F_i)$ in $C$, and maps $s^i_\ell, t^i_\ell$ between these, satisfying the two relations I wrote down yesterday (page 2) (and which of course have to be taken into account when defining $C_0$ to start with, I forgot to say before), and such that "standard"* fiber-products defined in terms of these data should exist in $C$, plus a bunch of maps between these fiber-products (in fact, it will suffice to give such maps with target among the $F_i$'s), satisfying certain relations embodied in the structure of the category $\mathbb{C}_\infty$. I am convinced that this bunch of maps (namely the maps stemming from arrows in $\mathbb{C}_\infty$) not only is infinite, but cannot either be generated in the obvious sense by a finite number, nor even by a finite number of infinite series of maps such as $\mathrm{k}^i_\ell$, $\mathrm{inv}_i$, $\overset{i}{\underset{\ell}{*}}$, the compatibility arrows in the pentagon, and the like. More precisely still, I am convinced that none of the functors $\mathbb{C}_n \to \mathbb{C}_{n+1}$ is an equivalence, which amounts to saying that the structures of increasing order form a strictly increasing sequence – at every step, there is actual extra structure added. This is perhaps evident beforehand to topologists in the know, but I confess that for the time being it isn't to me, in terms uniquely of the somewhat formal description I will make of the passage of $\mathbb{C}_n$ to $\mathbb{C}_{n+1}$. This theoretically is all that remains to be done, in order to achieve an explicit construction of the structure species of an $\infty$-category (besides the definition of standard fiber-products) – without having to get involved, still less lost, in the technical intricacies of ever messier diagrams to write down, with increasing order of the structures to be added...

*I'll drop the qualification "iterated" henceforth!

**6** In the outline of a method of construction for the structure species, there has not been any explicit mention so far of the topological motivation behind the whole approach, which could wrongly give the impression of being a purely algebraic one. However, topological considerations alone are giving me the clue both for the description of the so-called standard fiber products, and of the inductive step allowing to wind up from $\mathbb{C}_n$ to $\mathbb{C}_{n+1}$? The heuristics indeed of the present approach is simple enough, and suggested by the starting task, to define pertinent structure on the system of sets $F_i(X)$ of "homotopies" of arbitrary order, associated to an arbitrary topological space. In effect, the functors

*The topological model: hemispheres building up the (tentative) "universal $\infty$-(co)groupoid".*

$$X \mapsto F_i(X)$$

are representable by spaces $D_i$, which are easily seen to be $i$-disks. The source and target maps $s^i_\ell, t^i_\ell : F_i(X) \rightrightarrows F_{i-1}(X)$ are transposed to maps,

which I may denote by the same letters,

$$s^i_\ell, t^i_\ell : D_{i-1} \rightrightarrows D_i.$$

Handling around a little, one easily convinces oneself that all the main structural items on $F_*(X)$ which one is figuring out in succession, such as the degeneracy maps $k^i_\ell$, the inversion maps $\mathrm{inv}_i$, the composition $v \cdot u = v *_i u$ for $i$-objects, etc., are all transposed of similar maps which are defined between the cells $D_i$, or which go from such cells to certain spaces, deduced from these by gluing them together – the most evident example in this respect being the composition of paths, which is transposed of a map from the unit segment $I$ into $(I,1) \amalg_e (I,0)$, having preassigned values on the endpoints of $I$ (which correspond in fact to the images of the two maps $s^1_1, t^1_1 : D_0 = \text{one point} \rightrightarrows D_1 = I$). In a more suggestive way, we could say from this experiment that the family of discs $(D_i)_{i\in\mathbb{N}}$, together with the maps $s, t$ and a lot of extra structure which enters into the picture step by step, is what we would like to call a *co-$\infty$-groupoid in the category* (Top) of topological spaces (namely an $\infty$-groupoid in the dual category $(\mathrm{Top})^{\mathrm{op}}$), and that the structure of $\infty$-category on $F_*(X)$ we want to describe is the transform of this co-structure into an $\infty$-groupoid, by the contravariant functor from (Top) to (Sets) defined by $X$. The (iterated) amalgamated sums in (Top) which allow to glue together the various $D_i$'s using the $s$ and $t$ maps between them, namely the corresponding fibered products in $(\mathrm{Top})^{\mathrm{op}}$, are indeed transformed by the functor $h_X$ into fibered products of (Sets). The suggestion is, moreover, that if we view our co-structure in (Top) as a co-structure in the subcategory of Top, say $B_\infty$, whose objects are the cells $D_i$ and the amalgamated sums built up with these which step-wise enter into play, and whose arrows are all those arrows which are introduced step-wise to define the co-structure, and all compositions of these – that this should be *the universal structure* in the sense dual to the one we have been contemplating before; or what amount to the same, that the corresponding $\infty$-groupoid structure in the dual category $B^{\mathrm{op}}_\infty$ is "universal" – which means essentially that it is none other than the universal structure in the category $\mathbb{C}_\infty$ we are after. Whether or not this expectation will turn out to be correct (I believe it is*), we should be aware that, while the successive introduction of maps between the cells $D_i$ and their "standard" amalgamated sums (which we will define precisely below) depends at every stage on arbitrary choices, the categories $\mathbb{C}_n$ and their limit $\mathbb{C}_\infty$ do not depend on any of these choices; assuming the expectation is correct, this means that up to (unique) isomorphism, the category $B_\infty$ (and each of the categories $B_n$ of which it appears as the direct limit) is independent of those choices – the isomorphism between two such categories transforming any one choice made for the first, into the corresponding choice made for the second. Also, while this expectation was of course the crucial motivation leading to the explicit description of $\mathbb{C}_0$ and of the inductive step from $\mathbb{C}_n$ to $\mathbb{C}_{n+1}$, this description seems to me a reasonable one and in any case it makes a formal sense, quite independently of whether the expectation proves valid or not.

*25.2. But no longer now and I do not really care! Cf. p. 10.

**7** Before pursuing, it is time to give a more complete description of the primitive structure on $(D_i)$, as embodied by the maps $s, t$, which I will now denote by*

$$\varphi_i^+, \varphi_i^- : D_i \rightrightarrows D_{i+1}.$$

It appears that these maps are injective, that their images make up the boundary $S_i = \dot{D}_{i+1}$ of $D_{i+1}$, more specifically these images are just two "complementary" hemispheres in $S_i$, which I will denote by $S_i^+$ and $S_i^-$. The kernel of the pair $(\varphi_i^+, \varphi_i^-)$ is just $S_{i-1} = \dot{D}_i$, and the common restriction of the maps $\varphi_i^+, \varphi_i^-$ to $S_{i-1}$ is an isomorphism

$$S_{i-1} \simeq S_i^+ \cap S_i^-.$$

This $S_{i-1}$ in turn decomposes into the two hemispheres $S_{i-1}^+, S_{i-1}^-$, images of $D_{i-1}$. Replacing $D_{i+1}$ by $D_i$, we see that the $i$-cell $D_i$ is decomposed into a union of $2i+1$ closed cells, one being $D_i$ itself, the others being canonically isomorphic to the cells $S_j^+, S_j^-$ $(0 \leq j \leq i)$, images of $D_j \to D_n$ by the iterated morphisms

$$\varphi_{n,j}^+, \varphi_{n,j}^- : D_j \rightrightarrows D_n.$$

This is a cellular decomposition, corresponding to a partition of $D_n$ into $2n+1$ open cells $D_n$, $S_j^+ = \varphi_{n,j}^+(D_j)$, $S_j^- = \varphi_{n,j}^-(D_j)$. For any cell in this decomposition, the incident cells are exactly those of strictly smaller dimension.

When introducing the operation $\overset{n}{\underset{\ell}{*}}$ with $\ell = n - j$, it is seen that this corresponds to choosing a map

$$D_n \to (D_n, S_j^+) \amalg_{D_j} (D_n, S_j^-),$$

satisfying a certain condition *, expressing the formulas I wrote down yesterday for $s_1$ and $t_1$ of $u *_\ell v$ – the formulas translate into demanding that the restriction of the looked-for map of $D_n$ to its boundary $S_{n-1}$ should be a given map (given at any rate, for $\ell \geq 2$, in terms of the operation $*_{\ell-1}$, which explains the point I made that the $*_\ell$-structure is of order just above the $*_{\ell-1}$-structure, namely (inductively) is of order $\ell \ldots$). That the extension of this map of $S_{n-1}$ to $D_n$ does indeed exist, comes from the fact that the amalgamated sum on the right hand side is contractible for obvious reasons.

This gives a clue of what we should call "standard" amalgamated sums of the cells $D_i$. The first idea that comes to mind is that we should insist that the space considered should be contractible, excluding amalgamated sums therefore such as

$$\begin{tikzcd} & \bullet \arrow[dr] & \\ \bullet \arrow[ur] \arrow[rr] & & \bullet \end{tikzcd} \quad \text{or} \quad \begin{tikzcd} \bullet \arrow[r, bend left] \arrow[r, bend right] & \bullet \end{tikzcd}$$

which are circles. This formulation however has the inconvenience of not being directly expressed in combinatorial terms. The following

*Gluing hemispheres: the "standard amalgamations".*

*NB it is more natural to consider $\varphi^+$ as "target" and $\varphi^-$ as "source".

formulation, which has the advantage of being of combinatorial nature, is presumably equivalent to the former, and gives (I expect) a large enough notion of "standardness" to yield for the corresponding notion of $\infty$-category enough structure for whatever one will ever need. In any case, it is understood that the "amalgamated sum" (rather, finite $\varinjlim$) we are considering are of the most common type, when $X$ is the finite union of closed subsets $X_i$, with given isomorphisms

$$X \simeq D_{n(i)},$$

the intersection of any two of these $X_i \cap X_j$ being a union of closed cells both in $D_{n(i)}$ and in $D_{n(j)}$. (This implies in fact that it is either a closed cell in both, or the union of two closed cells of same dimension $m$ and hence isomorphic to $S_m$, a case which will be ruled out anyhow by the triviality condition which follows.[†]) The triviality or "standardness" condition is now expressed by demanding that the set of indices $I$ can be totally ordered, i.e., numbered in such a way that we get $X$ by successively "attaching" cells $D_{n(i)}$ to the space already constructed, $X(i-1)$, by a map from a sub-cell of $D_{n(i)}$, $S_j^\xi \to X(i-1)$ ($\xi \in \{\pm 1\}$), this map of course inducing an isomorphism, more precisely *the* standard isomorphism, $\varphi_{n(i)}^\xi : S_j^\xi \simeq D_j$ with $S_j^\xi$ one of the two corresponding cells $S_j^+, S_j^-$ in some $X_{i'} \simeq D_{n(i')}$. The dual translation of this, in terms of fiber products in a category $C$ endowed with objects $F_i$ ($i \in \mathbb{N}$) and maps $s_1^i, t_1^i$ between these, is clear: for a given set of indices $I$ and map $i \mapsto n(i) : I \to \mathbb{N}$, we consider a subobject of $\prod_{i \in I} D_{n(i)}$, which can be described by equality relations between iterated sources and targets of various components of $u = (u_i)_{i \in I}$ in $P$, the structure of the set of relations being such that $I$ can be numbers, from 1 to $N$ say, in such a way that we get in succession $N-1$ relations on the $N$ components $u_i$ respectively ($2 \le i \le N$), every relation being of the type $f(u_i) = g(u_i')$, with $f$ and $g$ being iterated source of target maps, and $i' < i$. (Whether source or target depending in obvious way on the two signs $\xi, \xi'$.)

[†]False, [picture of disk divided in three]

21.2.

**8** Returning to the amalgamated sum $X = \bigcup_i X_i$, the cellular decompositions of the components $X_i \simeq D_{n(i)}$ define a cellular decomposition of $X$, whose set of cells with incidence relation forms a finite ordered set $K$, finite union of a family of subsets $(K_i)_{i \in I}$, with given isomorphisms

*Description of the universal* primitive *structure.*

$$f_i : K_i \simeq \mathrm{J}_{n(i)} \quad (i \in I),$$

where for every index $n \in \mathbb{N}$, $\mathrm{J}_n$ denotes the ordered set of the $2n+1$ cells $S_j^\xi$ ($0 \le j \le n-1$, $\xi \in \{\pm 1\}$), $D_n$ of the pertinent cellular decomposition of $D_n$. We may without loss of generality assume there is no inclusion relation between the $K_i$, moreover the standardness condition described above readily translates into a condition on this structure $K$, $(K_i)_{i \in I}$, $(f_i)_{i \in I}$, and implies that for $i, i' \in I$, $K_i \to K_{i'}$, is a "closed" subset in the two ordered sets $K_i, K_{i'}$ (namely contains with any element $x$ the

elements smaller than $x$), and moreover isomorphic (for the induced order) to some $\mathrm{J}_n$. Thus the category $B_0$ can be viewed as the category of such "standard ordered sets" (with the extra structure on these just said), and the category $C_0$ can be defined most simply as the dual category $B_0^{\mathrm{op}}$. (NB the definition of morphisms in $B_0$ is clear I guess ...) I believe the category $B_0$ is stable under amalgamated sums $X \amalg_Z Y$, provided however we insist that the empty structure $K$ is *not* allowed – otherwise we have to restrict to amalgamated sums with $Z \neq \emptyset$. It seems finally more convenient to exclude the empty structure in $B_0$, i.e. to exclude the final element from $\mathbb{C}_0$, for the benefit of being able to state that $\mathbb{C}_0$ (and all categories $\mathbb{C}_n$*) are stable under amalgamated sums, and that the functors $\mathbb{C}_n \to C$ (and ultimately $\mathbb{C}_\infty \to C$) we are interested in are just those commuting to arbitrary binary amalgamated sums (without awkward reference to the objects $\boldsymbol{F}_i$ and the iterated source and target maps between them...).

*this is obviously false, see P.S. p.12.

**9** The category $\mathbb{C}_0$ being fairly well understood, it remains to complete the construction by the inductive step, passing from $\mathbb{C}_n$ to $\mathbb{C}_{n+1}$. The main properties I have in mind therefore, for the sequence of categories $\mathbb{C}_n$ and their limit $\mathbb{C}_\infty$, are the following two.

*The main inductive step: just add coherence arrows! The abridged story of an (inescapable and irrelevant) ambiguity*

(A) For any $K \in \mathrm{Ob}(\mathbb{C}_\infty)$ $(= \mathrm{Ob}(\mathbb{C}_0))$, and any two arrows in $\mathbb{C}_\infty$

$$f, g : K \rightrightarrows F_i,$$

with $i \in \mathbb{N}$, and such that either $i = 0$, or the equalities

$$s_1^i f = s_1^i g, \quad t_1^i f = t_1^i g \tag{1}$$

hold (case $i \geq 1$), there exists $h : K \to F_{i+1}$ such that

$$s_1^{i+1} h = f, \quad t_1^{i+1} h = g. \tag{2}$$

(B) For any $n \in \mathbb{N}$, the category $\mathbb{C}_{n+1}$ is deduced from $\mathbb{C}_n$ by keeping the same objects, and just adding new arrows $h$ as in (A), with $f, g$ arrows in $\mathbb{C}_n$.

The expression "deduces from" in (B) means that we are adding arrows $h : K \to F_i$ (each with preassigned source and target in $\mathbb{C}_n$), with as "new axioms" on the bunch of these uniquely the two relations (2) of (A), the category $\mathbb{C}_{n+1}$ being deduced from $\mathbb{C}_n$ in an obvious way, as the solution of a universal problem within the category of all categories where binary amalgamated products exist, and "maps" between these being functors which commute to those fibered products. In practical terms, the arrows of $\mathbb{C}_{n+1}$ are those deduced from the arrows in $\mathbb{C}_n$ and the "new" arrows $h$, by combining formal operations of composing arrows by $v \circ u$, and taking (binary) amalgamated products of arrows.*

*This has to be corrected – amalgamated sums exist in $\mathbb{C}$, only – and *those* should be respected.

NB Of course the condition (1) in (A) is necessary for the existence of an $h$ satisfying (2). That it is sufficient too can be viewed as an extremely strong, "universal" version of coherence conditions, concerning the various structures introduced on an $\infty$-groupoid. Intuitively, it means that whenever we have two ways of associating to a finite family $(u_i)_{i \in I}$

of objects of an $\infty$-groupoid, $u_i \in F_{n(i)}$, subjected to a standard set of relations on the $u_i$'s, an element of some $F_n$, in terms of the $\infty$-groupoid structure only, then we have automatically a "homotopy" between these built in in the very structure of the $\infty$-groupoid, provided it makes at all sense to ask for one (namely provided condition (1) holds if $n \geq 1$). I have the feeling moreover that conditions (A) and (B) (plus the relation $\mathbb{C}_\infty = \varinjlim \mathbb{C}_n$) is all what will be ever needed, when using the definition of the structure species, – plus of course the description of $\mathbb{C}_0$, and the implicit fact that the categories $\mathbb{C}_n$ are stable under binary fiber products and the inclusion functors commute to these.[†] Of course, the category which really interests us is $\mathbb{C}_\infty$, the description of the intermediate $\mathbb{C}_n$'s is merely technical – the main point is that there should exist an increasing sequence $(\mathbb{C}_n)$ of subcategories of $\mathbb{C}_\infty$, having the same objects (and the "same" fiber-products), such that $\mathbb{C}_\infty$ should be the limit (i.e., every arrow in $\mathbb{C}_\infty$ should belong to some $\mathbb{C}_n$), and such that the passage from $\mathbb{C}_n$ to $\mathbb{C}_{n+1}$ should satisfy (B). It is fairly obvious that these conditions alone do by no means characterize $\mathbb{C}_\infty$ up to equivalence, and still less the sequence of its subcategories $\mathbb{C}_n$. The point I wish to make though, before pursuing with a proposal of an explicit description, is that *this ambiguity is in the nature of things*. Roughly saying, two different mathematicians, working independently on the conceptual problem I had in mind, assming they both wind up with some explicit definition, will almost certainly get non-equivalent definitions – namely with non-equivalent categories of (set-valued, say) $\infty$-groupoids! And, secondly and as importantly, that *this ambiguity however is an irrelevant one*. To make this point a little clearer, I could say that a third mathematician, informed of the work of both, will readily think out a functor or rather a pair of functors, associating to any structure of Mr. X one of Mr. Y and conversely, in such a way that by composition of the two, we will associate to an $X$-structure ($T$ say) another $T'$, which will not be isomorphic to $T$ of course, but endowed with a canonical $\infty$-equivalence (in the sense of Mr. X) $T \underset{\infty}{\simeq} T'$, and the same on the Mr. Y side. Most probably, a fourth mathematician, faced with the same situation as the third, will get his own pair of functors to reconcile Mr. X and Mr. Y, which very probably won't be equivalent (I mean isomorphic) to the previous one. Here however, a fifth mathematician, informed about this new perplexity, will probably show that the two $Y$-structures $U$ and $U'$, associated by his two colleagues to an $X$-structure $T$, while not isomorphic alas, admit however a canonical $\infty$-equivalence between $U$, and $U'$ (in the sense of the $Y$-theory). I could go on with a sixth mathematician, confronted with the same perplexity as the previous one, who winds up with another $\infty$-equivalence between $U$ and $U'$ (without being informed of the work of the fifth), and a seventh reconciling them by discovering an $\infty$-equivalence between these equivalences. The story of course is infinite, I better stop with seven mathematicians, a fair number nowadays to allow themselves getting involved with foundational matters ... There should be a mathematical statement though resuming in finite terms this infinite story, but in order to write it down I guess a

[†] Inaccurate; see above

minimum amount of conceptual work, in the context of a given notion of $\infty$-groupoids satisfying the desiderata (A) and (B) should be done, and I am by no means sure I will go through this, not in this letter anyhow.

**10** Now in the long last the explicit description I promised of $\mathbb{C}_{n+1}$ in terms of $\mathbb{C}_n$. As a matter of fact, I have a handful to propose! One choice, about the widest I would think of, is: for every pair $(f,g)$ in $\mathbb{C}_n$ satisfying condition (1) of (A), add one new arrow $h$. To avoid set-theoretic difficulties though, we better first modify the definition of $\mathbb{C}_0$ so that the set of its objects should be in the universe we are working in, preferably even it be countable. Or else, and more reasonably, we will pick one $h$ for every isomorphism class of situations $(f,g)$ in $\mathbb{C}_n$. Another restriction to avoid too much redundancy – this was the first definition actually that flipped to my mind the day before yesterday – is to add a *new* $h$ only when there is no "old" one, namely in $\mathbb{C}_n$, serving the same purpose. Then it came to my mind that there is a lot of redundancy still, thus there would be already infinitely many operations standing for the single operation $v \overset{i}{\circ} u$ say, which could be viewed in effect in terms of an arbitrary $n$-sequence ($n \geq 2$) of "composable" $i$-objects $u_1 = u, u_2 = v, u_3, \ldots, u_n$. The natural way to meet this "objection" would be to restrict to pairs $(f,g)$ which cannot be factored non-trivially through another objects $K'$ as

*Cutting down redundancies – or: "l'embarras du choix".*

$$K \longrightarrow K' \underset{g'}{\overset{f'}{\rightrightarrows}} F_i \ .$$

But even with such restrictions, there remain a lot of redundancies – and this again seems to me in the nature of things, namely that there is no really natural, "most economic" way for achieving condition (A), by a stepwise construction meeting condition (B). For instance, in $\mathbb{C}_1$ already we will have not merely the compositions $v \overset{i}{\circ} u$, but at the same time simultaneous compositions

$$(*) \qquad u_n \circ u_{n-1} \circ \cdots \circ u_1$$

for "composable" sequences of $i$-objects ($i \geq 1$), without reducing this (as is customary) to iteration of the binary composition $v \circ u$. Of course using the binary composition, and more generally iteration of $n' - ary$ compositions with $n' < n$ (when $n \geq 3$), we get an impressive bunch of operations in the $n$ variables $u_1, \ldots, u_n$, serving the same purpose as (*). All these will be tied up by homotopies in the next step $\mathbb{C}_2$. We would like to think of this set of homotopies in $\mathbb{C}_2$ as a kind of "transitive system of isomorphisms" (of associativity), now the transitivity relations one is looking for will be replaced by homotopies again between compositions of homotopies, which will enter in the picture with $\mathbb{C}_3$, etc. Here the infinite story is exemplified by the more familiar situation of the two ways in which one could define a "$\otimes$-composition with associativity" in a category, starting either in terms of a binary operation, or with a bunch of $n$-ary operations – with, in

both cases the associativity isomorphisms being an essential part of the structure. Here again, while it is generally (and quite validly) felt that the two points of view are equivalent; and both have their advantages and their drawbacks, still it is not true, I believe, that the two categories of algebraic structures "category with associative $\otimes$-operation", using one or the other definition, are equivalent.* Here the story though of the relation between the two notions is a finite one, due to the fact that it is related to the notion of 2-categories or 2-groupoids, instead of $\infty$-groupoids as before...

*even not in the compoid??? context.

Thus I don't feel really like spending much energy in cutting down redundancies, but prefer working with a notion of $\infty$-groupoid which remains partly indeterminate, the main features being embodied in the conditions (A) and (B) and in the description of $\mathbb{C}_0$, without other specification.

[p. 10]

**11** One convenient way for constructing a category $\mathbb{C}_\infty$ would be to define for every $K, L \in \mathrm{Ob}(\mathbb{C}_0) = \mathrm{Ob}(B_0)$ the set $\mathrm{Hom}_\infty(K, L)$ as a subset of the set $\mathrm{Hom}(|L|, |K|)$ of continuous maps between the geometric realizations of $L$ and $K$ in terms of gluing together cells $D_i$, the composition of arrows in $\mathbb{C}_\infty$ being just composition of maps. This amounts to defining $\mathbb{C}_\infty$ as the dual of a category $\mathbb{B}_\infty$ of topological descriptions. It will be sufficient to define for every cell $D_n$ and every subset $\mathrm{Hom}_\infty(D_n, |K|)$ of $\mathrm{Hom}(D_n, |K|)$, satisfying the two conditions:

*Returning to the topological model (the canonical functor from spaces to "$\infty$-groupoids").*

(a) stability by compositions $D_n \to |K| \to |K'|$, where $K \to K'$ is an "allowable" continuous map, namely subjected only to the condition that its restriction to any standard subcell $D_{n'} \subset |K|$ is again "allowable", i.e., in $\mathrm{Hom}_\infty$.

(b) Any "allowable" map $S_n \to |K|$ (i.e., whose restrictions to $S_n^+$ and $S_n^-$ are allowable) extends to an allowable map $D_{n+1} \to |K|$.

Condition (a) merely ensures stability of allowable maps under composition, and the fact that $\mathbb{B}_\infty$ (endowed with the allowable maps as morphisms) has the correct binary amalgamated sums, whereas (b) expresses condition (a) on $\mathbb{C}_\infty$. These conditions are satisfied when we take as $\mathrm{Hom}_\infty$ subsets defined by tameness conditions (such as piecewise linear for suitable piecewise linear structure on the $D_n$'s, or differentiable, etc.). The condition (b) however is of a subtler nature in the topological interpretation and surely not met by such sweeping tameness requirements only! Finally, the question as to whether we can actually in this way describe an "acceptable" category $\mathbb{C}_\infty$, by defining sets $\mathrm{Hom}_\infty$, namely describing $\mathbb{C}_\infty$ in terms of $\mathbb{B}_\infty$, seems rather subsidiary after all. We may think of course of constructing stepwise $\mathbb{B}_\infty$ via subcategories $\mathbb{B}_n$, by adding stepwise new arrows in order to meet condition (b), thus paraphrasing condition (B) for passage from $\mathbb{C}_n$ to $\mathbb{C}_{n+1}$ – but it is by no means clear that when passing to the category $\mathbb{B}_{n+1}$ by composing maps of $\mathbb{B}_n$ and "new" ones, and using amalgamated sums too, there might not be some undesirable extra relations in $\mathbb{B}_{n+1}$, coming from the topological interpretation of the arrows in $\mathbb{C}_{n+1}$ as maps. To say it differently, universal algebra furnishes us readily with

an acceptable sequence of categories $\mathbb{C}_n$ and hence $\mathbb{C}_\infty$, and by the universal properties of the $\mathbb{C}_n$ in terms of $\mathbb{C}_{n+1}$, we readily get (using arbitrary choices) a contravariant functor $K \mapsto |K|$ from $\mathbb{C}_\infty$ to the category of topological spaces (i.e., a co-$\infty$-groupoid in (Top)), but it is by no means clear that this functor is faithful – and it doesn't really matter after all!

**12** I think I really better stop now, except for one last comment. The construction of a co-$\infty$-groupoid in (Top), giving rise to the fundamental functor

*About replacing spaces by objects of a "model category".*

$$(\mathrm{Top}) \longrightarrow (\infty\text{-groupoids}),$$

generalizes, as I already alluded to earlier, to the case when (Top) is replaced by an arbitrary "model category" $M$ in your sense. Here however the choices occur not only stepwise for the primary, secondary, ternary etc. structures, but already for the primitive structures, namely by choice of objects $D_i$ ($i \in \mathbb{N}$) in $M$, and source and target maps $D_i \rightrightarrows D_{i+1}$. These choices can be made inductively, by choosing first for $D_0$ the final object, or more generally any object which is fibrant and trivial (over the final objects), $D_{-1}$ being the initial object, and defining further $S_0 = D_0 \amalg_{D_{-1}} D_0 = D_0 \amalg D_0$ with obvious maps $\psi_0^+, \psi_0^- : D_0 \to S_0$, and then, if everything is constructed up to $D_n$ and $S_n = (D_n, \varphi_{n-1}^+) \amalg_{D_{n-1}} (D_n, \varphi_{n-1}^-)$, defining $D_{n+1}$ as any fibrant and trivial object together with a cofibrant map

$$S_n \to D_{n+1},$$

and $\varphi_n^+, \varphi_n^-$ as the compositions of the latter with $\psi_n^+, \psi_n^- : D_n \rightrightarrows S_n$. Using this and amalgamated sums in $M$, we get our functor

$$\mathbb{B}_0 = \mathbb{C}_0^{\mathrm{op}} \to M, \quad K \mapsto |K|_M,$$

commuting with amalgamated sums, which we can extend stepwise through the $\mathbb{C}_n^{\mathrm{op}}$'s to a functor $\mathbb{B}_\infty = \mathbb{C}_\infty^{\mathrm{op}} \to M$, provided we know that the objects $|K|_M$ of $M$ ($K \in \mathrm{Ob}\,\mathbb{C}_0$) obtained by "standard" gluing of the $D_n$'s in $M$, are again fibrant and trivial – and I hope indeed that your axioms imply that, via, say, that if $Z \to X$ and $Z \to Y$ are cofibrant and $X, Y, Z$ are fibrant and trivial, then $X \amalg_Z Y$ is fibrant and trivial...

Among the things to be checked is of course that when we localize the category of $\infty$-groupoids with respect to morphisms which are "weak equivalences" in a rather obvious sense (NB the definition of the $\Pi_i$'s of an $\infty$-groupoid is practically trivial!), we get a category equivalent to the usual homotopy category (Hot). Thus we get a composed functor

$$M \to (\infty\text{-groupoids}) \to (\mathrm{Hot}),$$

as announced. I have some intuitive feeling of what this functor stands for, at least when $M$ is say the category of semisimplicial sheaves, or (more or less equivalently) of $n$-gerbes or $\infty$-gerbes on a given topos: namely it should correspond to the operation of "integration" or "sections" for $n$-gerbes (more generally for $n$-stacks) over a topos – which is

indeed *the* basic operation (embodying non-commutative cohomology objects of the topos) in "non-commutative homological algebra".

I guess that's about it for today. It's getting late and time to go to bed! Good night.

22.2.1983

**13** It seems I can't help pursuing further the reflection I started with this letter! First I would like to come back upon terminology. Maybe to give the name of $n$-groupoids and $\infty$-groupoids to the objects I was after is not proper, for two reasons: a) it conflicts with a standing terminology, applying to structure species which are frequently met and deserve names of their own, even if they turn out to be too restrictive kind of objects for the use I am having in mind – so why not keep the terminology already in use, especially for two-groupoids, which is pretty well suited after all; b) the structure species I have in mind is not really a very well determined one, it depends as we saw on choices, without any one choice looking convincingly better than the others – so it would be a mess to give an unqualified name to such structure, depending on the choice of a certain category $\mathbb{C}_\infty = \mathbb{C}$. I have been thinking of the terminology $n$-stack and $\infty$-stack (stack = "champ" in French), a name introduced in Giraud's book (he was restricting to champs = 1-champs), which over a topos reduced to a one-point space reduces in his case to the usual notion of a category, i.e., 1-category. Here of course we are thinking of "stacks of groupoids" rather than arbitrary stacks, which I would like to call (for arbitrary order $n \in \mathbb{N}$ or $n = \infty$) $n$-Gr-stack – suggesting evident ties with the notion of Gr-categories, we should say Gr-1-categories, of Mme Hoang Xuan Sinh. One advantage of the name "stack" is that the use it had so far spontaneously suggests the extension of these notions to the corresponding notions over an arbitrary topos, which of course is what I am after ultimately. Of course, when an ambiguity is possible, we should speak of $n$-$\mathbb{C}$-stacks – the reference to $\mathbb{C}$ should make superfluous the "Gr" specification. Thus $n$-$\mathbb{C}$-stacks are essentially the same as $n$-$\mathbb{C}$-stacks over the final topos, i.e., over a one point space. When both $\mathbb{C}$ and "Gr" are understood in a given context, we will use the terminology $n$-stack simply, or even "stack" when $n$ is fixed throughout. Thus it will occur that in certain contexts "stack" will just mean a usual groupoid, in others it will mean just a category, but when $n = 2$ it will not mean a usual 2-groupoid, but something more general, defined in terms of $\mathbb{C}$.

*An urgent reflection on proper names: "Stacks" and "coherators".*

The categories $\mathbb{C} = \mathbb{C}_\infty$ described before merit a name too – I would like to call them "coherators" ("cohéreurs" in French). This name is meant to suggest that $\mathbb{C}$ embodies a hierarchy of coherence relations, more accurately of coherence "homotopies". When dealing with stacks, the term *$i$-homotopies* (rather than $i$-objects or $i$-arrows) for the elements of the $i^{\text{th}}$ component $F_i$ of a stack seems to me the most suggestive – they will of course be denoted graphically by arrows, such as $h : f \to g$ in the formulation of (A) yesterday. More specifically, I will call coherator any category *equivalent* to a category $\mathbb{C}_\infty$ as constructed before. Thus

a coherator is stable under binary fiber products,[*] moreover the $F_i$ are recovered up to isomorphism as the indecomposable elements of $\mathbb{C}$ with respect to amalgamation. However, in a category $\mathbb{C}_\infty$, the objects $\boldsymbol{F}_i$ have non-trivial automorphisms – namely the "duality involutions" and their compositions (the group of automorphisms of $\boldsymbol{F}_i$ should turn out to be canonically isomorphic to $(\pm 1)^i$),[†] in other words by the mere category structure of a coherator we will not be able to recover the objects $\boldsymbol{F}_i$ in $\mathbb{C}$ up to unique isomorphism. Therefore, in the structure of a coherator should be included, too, the choice of the basic indecomposable objects $\boldsymbol{F}_i$ (one in each isomorphism class), and moreover the arrows $\mathrm{s}_1^i, \mathrm{t}_1^i : \boldsymbol{F}_i \rightrightarrows \boldsymbol{F}_{i-1}$ for $i \geq 1$ (a priori, only the *pair* $(\mathrm{s}_1^i, \mathrm{t}_1^i)$ can be described intrinsically in terms of the category structure of $\mathbb{C}$, once $\boldsymbol{F}_i$ and $\boldsymbol{F}_{i-1}$ are chosen...). But it now occurs to me that we don't have to put in this extra structure after all – while the $\boldsymbol{F}_i$'s separately do have automorphisms, the system of objects $(\boldsymbol{F}_i)_{i\in\mathbb{N}}$ and of the maps $(\mathrm{s}_1^i)_{i\geq 1}$ and $(\mathrm{t}_1^i)_{i\geq 1}$ has only the trivial automorphism (all this of course is heuristics, I didn't really prove anything – but the structure of the full subcategory of a $\mathbb{C}_\infty$ formed by the objects $\boldsymbol{F}_i$ seems pretty obvious...). To finish getting convinced that the mere category structure of a coherator includes already all other relevant structure, we should still describe a suitable intrinsic filtration by subcategories $\mathbb{C}_n$. We define the $\mathbb{C}_n$ inductively, $\mathbb{C}_0$ being the "primitive structure" (the arrows are those deducible from the source and target arrows by composition and fiber products), and $\mathbb{C}_{n+1}$ being defined in terms of $\mathbb{C}_n$ as follows: add to $\mathrm{Fl}\,\mathbb{C}_n$ all arrows of $\mathbb{C}$ of the type $h : K \to \boldsymbol{F}_i$ $(i \geq 1)$ such that $\mathrm{s}_1^i h$ and $\mathrm{t}_1^i h$ are in $\mathbb{C}_n$, and the arrows deduced from the bunch obtained by composition and fibered products.[‡] In view of these constructions, it would be an easy exercise to give an intrinsic characterization of a coherator, as a category satisfying certain internal properties.

I was a little rash right now when making assertions about the structure of the group of automorphisms of $\boldsymbol{F}_i$ – I forgot that two days ago I pointed out to myself that even the basic operation $\mathrm{inv}_i$ upon $\boldsymbol{F}_i$ need not even be involutions![§] However, I just checked that if in the inductive construction of coherators $\mathbb{C}_\infty$ given yesterday, we insist on the most trivial irredundancy condition (namely that we don't add a "new" homotopy $h : f \to g$ when there is already an old one), then any morphism $h : \boldsymbol{F}_i \to \boldsymbol{F}_i$ such that $\mathrm{s}f = \mathrm{s}$ and $\mathrm{t}f = \mathrm{t}$, is the identity – and that implies inductively that an automorphism of the system of $\boldsymbol{F}_i$'s related by the source and target maps $\mathrm{s}_1^i, \mathrm{t}_1^i$ is the identity. Thus it is correct after all, it seems, that the category structure of a coherator implies all other structure relevant to us.[¶]

I do believe that the description given so far of what I mean by a coherator, namely something acting like a kind of pattern in order to define a corresponding notion of "stacks" (which in turn should be the basic coefficient objects in non-commutative homological algebra, as well as a convenient description of homotopy types) embodies some of the essential features of the theory still in embryo that wants to be developed. It is quite possible of course that some features are lacking still, for instance that some extra conditions have to be imposed

[*] false, see PS

[†] false, see below

[‡] This may give however *too large* a category $\mathbb{C}_{n+1}$.

[§] Nor even automorphisms

[¶] Maybe false; it is safer to give moreover the subcategory $\mathbb{C}_0$.

upon $\mathbb{C}$, possibly of a very different nature from mere irredundancy conditions (which, I feel, are kind of irrelevant in this set-up). Only by pushing ahead and working out at least in outline the main aspects of the formalism of stacks, will it become clear whether or not extra conditions on $\mathbb{C}$ are needed. I would like at least to make a commented list of these main aspects, and possibly do some heuristic pondering on some of these in the stride, or afterwards. For today it seems a little late though – I have been pretty busy with non-mathematical work most part of the day, and the next two days I'll be busy at the university. Thus I guess I'll send off this letter tomorrow, and send you later an elaboration (presumably much in the style of this unending letter) if you are interested. In any case I would appreciate any comments you make – that's why I have been writing you after all! I will probably send copies to Ronnie Brown, Luc Illusie and Jean Giraud, in case they should be interested (I guess at least Ronnie Brown is). Maybe the theory is going to take off after all, in the long last!

Very cordially yours

PS (25.2.) I noticed a rather silly mistake in the notes of two days ago, when stating that the categories $\mathbb{C}_n$ admit fiber products: what is true is that the category $\mathbb{C}_0$ has fiber products (by construction, practically), and that these are fiber products also in the categories $\mathbb{C}_n$ (by construction equally), i.e., that the inclusion functors $\mathbb{C}_0 \to \mathbb{C}_n$ commute to fiber products. Stacks in a category $C$ correspond to functors $\mathbb{C}_\infty \to C$ whose restriction to $\mathbb{C}_0$ commutes with fiber products. I carried the mistake along in the yesterday notes – it doesn't really change anything substantially. I will have to come back anyhow upon the basic notion of a coherator...