Buffalo May 19, 1973

Dear Finn Knudsen,

Mumford sent me your notes on the determinant of perfect complexes, asking me to write you some comments, if I have any. Indeed I do have several - except for the obvious one that it is nice to have written up with details at least *one* full construction of that damn functor! I did not enter into the technicalities of your construction, which perhaps will allow to get a better comprehension of the main result itself. The main trouble with your presentation seems to me that the bare statement of the main result looks rather mysterious and not "natural" at all, despite your claim on page 3b! The mysterious character is of course included in the alambicated sign of definition 1.1. Here two types of criticism come to mind:

1) The sign looks complicated - are there not simpler sign conventions for getting a nice theory of $\det^*$ and its variance? It seems to me that Deligne wrote down a system that really did look natural at every stage - however he never wrote down the explicit construction, as far as I know, and the chap who had undertaken to do so, gave up in disgust after a year or two of letting the question lie around and rot!

2) Even granted that your conventions are as simple or simpler than other ones, the very fact that they are so alambicated and technical calls for an elucidation, somewhat of the type you give on page 3b with those ϵ_i's. That is one would like to *define* first what any theory of $\det^*$ should be (with conventions of sign as yet unspecified), stating say something like a *uniqueness theorem* for every given system of signs chosen for canonical isomorphisms, and moreover *characterizing* those systems of sign conventions which allow for an existence theorem - which will include the existence of at least one such system of signs. If one has good insight into all of them, it will be a matter of taste and convenience for the individual mathematician (or the situation he has to deal with in any instance) to make his own choice!

A second point is the introduction of such evidently superfluous assumptions like working on Noetherian (!) schemes, whereas the construction is clearly so general as to work, say, over any ringed space and even ringed topos - and of course it will be needed in this generality, for instance on analytic spaces, or on schemes with groups of automorphisms acting, etc. Its just a question of some slight extra care in the writing up. It is clear in any case that the question reduces to defining $\det^*$ for strictly perfect complexes (i.e. which are free of finite type in every degree), and for homotopy classes of homotopy equivalences between such complexes, as well as for

short exact sequences of such complexes. (NB! One may wish to deal, more generally, in the Illusie spirit, with strictly perfect complexes filtered - by a filtration which is finite but possibly not of level two - by sub-complexes with strictly perfect quotients.) Now this allows to restate the whole thing in a more general setting, which could make the theory more transparent, namely:
An additive category $\mathcal{C}$ (say free (or projective) modules of finite type over a commutative ring A) is given, as well as a category $\mathcal{P}$ which is a groupoid, endowed with an operation $\otimes$ together with associativity, unity and commutativity data, satisfying the usual compatibilities (see for instance Saavedra's thesis in Springer's lecture notes) and with all objects "invertible". In the example for $\mathcal{C}$, we take for $\mathcal{P}$ invertible $\mathbb{Z}$-graded modules over A, with tensor product, the commutative law $L \otimes L' \simeq L' \otimes L$ involving the Koszul sign $(-1)^{dd'}$ where d and d' are the degrees of L and L' respectively. We are interested in functors (or a given functor) $f : (\mathcal{C}, \mathrm{isom}) \to \mathcal{P}$, together with a functorial isomorphism $f(M+N) \simeq f(M) \otimes f(N)$, compatible with the associativity and commutativity data (cf. Saavedra for this notion of a $\otimes$); for instance, in the example chosen, we take $f(M) = \det^*(M)$, the determinant module where $*$ stands for the degree which we put on the determinant module (our convention will be to put the degree equal to the rank of M, which will imply that our functor is indeed compatible with the commutativity data). It can be shown (this was done by a North Vietnamese mathematician, Sinh Hoang Xuan) that given $\mathcal{C}$ (indeed any associative and commutative $\otimes$-category would do), there exists a universal way of sending $\mathcal{C}$ to $\mathcal{P}$ as above - in the case considered, this category can be called the category of "stable" projective modules over A, and its main invariants (isomorphism classes of objects, and automorphisms of the unit object) are just the invariants $\mathrm{K}^0(A)$ and $\mathrm{K}^1(A)$ of myself and Dieudonné-Bass; but this existence of a universal situation is irrelevant for the technical problem to come. Now consider the category $K = \mathrm{K}^b(\mathcal{C})$, of bounded complexes of $\mathcal{C}$, up to homotopy. It is a triangulated category [2], and as such we can

[2]Be careful that one has to take the term "triangulated category" in a slightly more precise sense than in Verdier's notes, the "category of triangles" being something more precise than a mere category of distinguished diagrams in K. We have a functor from the former to the latter, but it is not even a faithful one. (Illusie's treatment in terms of filtered complexes, in his Springer lecture notes, is a good reference) It is with respect to the category of "true" triangles only that the isomorphism $g(M) \simeq g(M') \otimes g(M'')$ will be functorial. For instance, if we have an *automorphism* of a triangle, inducing u, u', u'' upon M, M' and M'', then functoriality is expressed by the relation $\det u = \det u' \det u''$ (which implies, replacing u by $\mathrm{id} + tu$, t an indeterminate, that $\mathrm{Tr} u = \mathrm{Tr} u' + \mathrm{Tr} u''$) but this relation may become *false* if we are not careful to take automorphisms of true triangles, instead of taking mere automorphisms of diagrams.

define the notion of a $\otimes$-functor from K into $\mathcal{P}$; it's first of all a $\otimes$-functor for the additive structure of K (the internal composition of K being $\oplus$), but with moreover an extra structure consisting giving isomorphisms $g(M) \simeq g(M') \otimes g(M'')$ whenever we have an exact triangle $M' \to M \to M'' \to M'$. This should of course satisfy various conditions, such as functoriality with respect to the triangle [1], case of split exact triangle ($M = M' \oplus M''$), case of the triangle obtained by completing a quasi-isomorphism $M' \to M$, and possibly also a condition of compatibility in the case of an exact triangle of triangles. (I guess Deligne wrote down the reasonable axioms some day; it may be more convenient to work with the filtered K-categories of Illusie, using of course finite filtrations that split in the present context). Of course if we have such a $g : K \to \mathcal{P}$, taking its "restriction" to $\mathcal{C}$ we get an $f : \mathcal{C} \to \mathcal{P}$. The beautiful statement to prove would then be that conversely, every given f extends, uniquely up to isomorphism, to a g, in other terms, that the restriction functor from the category of g's to the category of f's is an equivalence. The whole care, for such a statement, will of course be to give the right set of "sign conventions" for defining admissible g's (that is compatibilities between the two or three structures on the set of $g(M)$'s- which in fact all can be reduced to giving the isomorphisms attached to exact triangles). In this general context, the group of signs ± 1 is replaced by the subgroup of elements of order 2 of the group $\mathrm{K}^1(\mathcal{P}) = \mathrm{Aut}(1_{\mathcal{P}})$ (which is always a commutative group). The "sign map" $n \to (-1)^n$ from the group of degrees to the group of signs is replaced here by a canonical map $\mathrm{K}^0(\mathcal{P})(=$ group of isomorphism classe of $\mathcal{P}) \to \mathrm{K}^1(\mathcal{P})$, associating to every L in $\mathcal{P}$ the symmetry automorphism of $L \otimes L$ (viewed as coming from an automorphism of the unit object by tensoring with $L \otimes L$). What puzzles me a little is that apparently, you have not been able to define g in terms intrinsic to the triangulated category $K = \mathrm{K}^b(\mathcal{C})$ - the signs you introduce in 1.1 do depend on the actual complexes, not only on their homotopy classes. I guess the whole trouble comes from the order in which we write any given tensor product in $\mathcal{P}$, in describing $\det^*(M^\bullet)$ we had to choose such an order rather arbitrarily, and it is passing from one such to another that involves "signs".

If $\mathcal{C}$ is an *abelian* category, there should be a variant of the previous theory, putting in relations on the $\otimes$-functors $f : \mathcal{C} \to \mathcal{P}$ together with the extra structure of isomorphisms $f(M) \simeq f(M') \otimes f(M'')$ for all short exact sequences $0 \to M' \to M \to M'' \to 0$ satisfying a few axioms, and $\otimes$-functors $g : \mathrm{D}^b(\mathcal{C}) \to \mathcal{P}$. There should also be higher dimensional analogous, involving $\mathcal{P}$'s that are n-categories instead of mere 1-categories, and hence involving (implicitly at least) the higher K-invariants $\mathrm{K}^i(\mathcal{C})$ $(i \geq 0)$. But of course, first of all the case of the relation between $\mathcal{C}$ and $\mathrm{K}^b(\mathcal{C})$ in the simplest case should be elucidated!

I am finishing this letter at the forum where I have no typewriter. I hope you can read the handwriting!

Best wishes A. Grothendieck