Letter of A. Grothendieck to J. Coates[1]

6.1.1966

Dear Coates,

Here a few more comments to my talk on the conjectures. The following proposition shows that the conjecture $C_\ell(X)$ is independent of the chosen polarisation, and has also some extra interest, in showing the part played by the fact that $\mathrm{H}^i(X)$ should be "motive-theoretically" isomorphic to its natural dual $\mathrm{H}^{2n-i}(X)$ (as usual, I drop the twist for simplicity).

Proposition. — *The condition $C_\ell(X)$ is equivalent also to each of the following conditions:*

a) $D_\ell(X)$ holds, and for every $i < n$, there exists an isomorphism $\mathrm{H}^{2n-i}(X) \longrightarrow \mathrm{H}^i(X)$ which is algebraic (i.e. induced by an algebraic correspondence class; we do not make any assertion on what it induces in degrees different from $2n-i$).

b) For every endomorphism $\mathrm{H}^i(X) \longrightarrow \mathrm{H}^i(X)$ which is algebraic, the coefficients of the characteristic polynomial are rational, and for every $i < n$, there exists an isomorphism $\mathrm{H}^{2n-i}(X) \longrightarrow \mathrm{H}^i(X)$ which is algebraic.

Proof. — I sketched already how $D_\ell(X)$ implies the fact that for an algebraic endomorphism of $\mathrm{H}^i(X)$, the coefficients of the characteristic polynomial are rational numbers, therefore we know that a) implies b), and of course $C_\ell(X)$ implies a). It remains to prove that b) implies $C_\ell(X)$. Let $u : \mathrm{H}^{2n-i}(X) \longrightarrow \mathrm{H}^i(X)$ be the given isomorphism which is algebraic, and $v : \mathrm{H}^i(X) \longrightarrow \mathrm{H}^{2n-i}(X)$ is an algebraic isomorphism in the opposite direction, induced by L_X^{n-i}. Then $uv = w$ is an automorphism of $\mathrm{H}^i(X)$ which is algebraic, and the Hamilton-Cayley formula $u^h - \sigma_1(w)u^{h-1} + ... + (-1)^b\sigma_b(w) = 0$

1. This text had been transcribed by Mateo Carmona

(where the $\sigma_i(w)$ are the coefficients of the characteristic polynomial of w) such that w^{-1} is a linear combination of the w^i, with coefficients of the type $+/-\sigma_i(w)/\sigma_b(w)$ (N.B. $b = \operatorname{rank} \mathrm{H}^i$). The assumption implies that these coefficients are rational, which implies that w^{-1} is algebraic, and so is $w^{-1}u = v^{-1}$, which was to be proved.

N.B. In characteristic 0, the statement simplifies to: $C(X)$ equivalent to the existence of algebraic isomorphisms $\mathrm{H}^{2n-i}(X) \longrightarrow \mathrm{H}^i(X)$, (as the preliminary in b) is then automatically satisfied). Maybe with some extra care this can be proved too in arbitrary characteristics.

Corollary. — *Assume X and X' satisfy condition C_ℓ, and let $u : \mathrm{H}^i(X) \longrightarrow \mathrm{H}^{i+2D}(X') \longrightarrow \mathrm{H}^i(X)$ ($D \in \mathbf{Z}$) be an isomorphism which is algebraic. Then u^{-1} is algebraic.*

Indeed, the two spaces can be identified "algebraically" (both directions!) to their dual, so that the transpose of u can be viewed as an isomorphism $u' : \mathrm{H}^{i+2D}(X') \longrightarrow \mathrm{H}^i(X)$. Thus $u'u$ is an algebraic automorphism w of $\mathrm{H}^i(X)$, and by the previous argument we see that w^{-1} is algebraic, hence so is $u^{-1} = w^{-1}u'$.

As a consequence , we see that if $x \in \mathrm{H}^i(X)$ is such that $u(x)$ is algebraic (i being now assumed to be even), than so is x. The same result should hold in fact if u is a monomorphism, the reason being that in this case there should exists a left-inverse which is algebraic; this exists indeed in a case like $\mathrm{H}^{n-1}(X) \longrightarrow \mathrm{H}^{n-1}(Y)$ (where we take the left inverse $\Lambda_X\varphi_*$). But to get it in general, it seems w need moreover the Hodge index relation. (The complete yoga then being that we have the category of motives which is semi-simple!). Without speaking of motives, and staying down on earth, it would be nice to explain in the notes that $C(X)$ together with the index relation $I(X \times X)$ implies that the ring of correspondences classes for X is semi-simple, and how one deduces from this the existence of left and right inverses as looked for above.

This could be given in an extra paragraph (which I did not really touch upon in the talk), containing also the deduction of the Weil conjectures from the conjectures C and A.

A last and rather trivial remark is the following. Let's introduce variant $A'_\ell(X)$ and $A''_\ell(X)$ as follows:

$A'_\ell(X)$: if $2i \leq n-1$, any element x of $\mathrm{H}^i(X)$ whose image in $\mathrm{H}^i(Y)$ is algebraic, is algebraic.

$A''_\ell(X)$: if $2i \geq n-1$, any algebraic element of $\mathrm{H}^{i+2}(X)$ is the image of an algebraic element of $\mathrm{H}^i(Y)$.

Let us consider also the specifications $A'_\ell(X)^\circ$ and $A''_\ell(X)^\circ$, where we restrict to the [] dimensions $2i = n-1$ if n odd, $2i = n-2$ if n even. All these conditions are in the nature of "weak" Lefschetz relations, and they are trivially implied by $A_\ell(X)$ resp. $C_\ell(X)$ (in the first case, applying φ we see that $L_X X$ is algebraic; in the second, we take $y = \Lambda_Y \varphi^+(x)$). The remark then is that these pretendently "weak" variants in fact imply the full Lefschetz relations for algebraic cycles, namely:

Proposition. — *$C_\ell(X)$ is equivalent to the conjunction $C_\ell(Y)+A_\ell(X\times X)^\circ + A''_\ell(X\times X)^\circ$, hence (by induction) also to the conjunction of the conditions A'°_ℓ and A''°_ℓ for all of the varieties $X\times X$, $Y\times Y$, $Z\times Z$,.... Analogous statement with $X\times Y$, $Y\times Z$ etc instead of $X\times X$, $Y\times Y$ etc.*

This comes from the remark that $A_\ell(X)^\circ$ follows from the conjunction of $A'_\ell(X)^\circ$ and $A''_\ell(X)^\circ$, as one sees by decomposing $L^2_X : \mathrm{H}^{2m-2}(X) \longrightarrow \mathrm{H}^{2m+2}(X)$ into $\mathrm{H}^{2m+2}(X) \xrightarrow{\varphi^k} \mathrm{H}^{2m+2}(Y) \xrightarrow{\varphi\alpha} \mathrm{H}^{2m}(X) \xrightarrow{L_X} \mathrm{H}^{2m+2}(X)$ if dim $X = 2m$ is even, and $\mathrm{H}^{2m+1-1}(X) \longrightarrow \mathrm{H}^{2m+1+1}$ into $\mathrm{H}^{2m}(X) \xrightarrow{\varphi^*} \mathrm{H}^{2m}(Y) \xrightarrow{\varphi_*} \mathrm{H}^{2m+2}(X)$ if dim $X = 2m+1$ is odd.

Sincerely yours
