Letter of A. Grothendieck to L. Breen[1]

Villecun 17.2.75

Dear Larry,

Here is an afterthought to "une lettre-fleuve" on the yoga of homotopy. As you doubtless know, to a topos X one associates canonically a pro-simplicial set, and so in a convenient sense a "pro-homotopy type". When X is "locally homotopically trivial", the associated pro-object is essentially constant as a pro-object in the homotopy category, and so X defines, in the usual homotopy category, an object which is the "homotopy type". Similarly, if X is "locally homotopically trivial in dim $\leq n$", it defines an ordinary homotopy type, but "truncated in dim $\leq n$" - this is a familiar construction for $n = 0$ or 1, even among those like me who know hardly any homotopy theory!

These constructions are functorial in X. Moreover, if $f : X \longrightarrow Y$ is a morphism of topoi, Artin-Mazur have given a *cohomological* condition which is necessary and sufficient for f to be a "homotopy equivalence in dim $\leq n$": it is that $\mathrm{H}^i(Y,F) \xrightarrow{\sim} \mathrm{H}^i(X, f^*(F))$ for $i \leq n$, and all *locally constant* sheaves of groups F on Y, allowing for $i \leq 1$ that F be non-commutative. This criterion, in terms of "locally constant" n-gerbes F on Y, can be interpreted as the condition that $F(Y) \longrightarrow F(X)$ is an n-equivalence for all such F and $i \leq n$. It is certainly true that this is equivalent to the following criterion:

(A) For every "locally constant" n-stack F on Y, the n-functor $F(Y) \longrightarrow f^*(F)(X)$ is an n-equivalence;

or again

[1]This text had been transcribed by Mateo Carmona

(B) The n-functor $F \longrightarrow f^*(F)$ which sends the n-category of locally constant $(n-1)$-stacks in Y to that of locally constant $(n-1)$-stacks on X, is an n-equivalence.

In other terms, the construction on a topos X which one can make in terms of $(n-1)$-stacks which are *locally* constant, depend only on its "n-truncated pro-homotopy type", and define it. In the case where X is locally homotopically trivial in dim $\leq n$, and so defines a n-truncated ordinary homotopy type, one can interpret these last as an n-groupoid C_n, (defined up to n-equivalence). In terms of these

(C) The $(n-1)$-stacks on X should be able to be identified with the n-functors from the category C_n n-category $(n-1)-\mathrm{Cat}$ of all $(n-1)$-categories.

In the case $n = 1$, this is nothing other than the Poincaré theory of the classification of coverings of X in terms of the "fundamental groupoid" C_1 of X. By extension, C_n merits the name *fundamental n-groupoid* of X, which I propose to write $\Pi_n(X)$. Knowledge of this includes knowledge of the $\pi_i(X)$ $(0 \geq i \geq n)$ and the Postnikoff invariants of all orders up to $\mathrm{H}^{n+1}(\Pi_{n-1}(X), \pi_n)$.

In the case of an arbitrary topos X, not necessarily locally homotopically trivial in dim $\leq n$, one hopes to be able to interpret the $(n-1)$-stacks which are locally constant on X in terms of a $\Pi_n(X)$ which will be a pro-n-groupoid. This has been done, more or less, for $n = 1$ (at least for connected X); the case where X is the étale topos of a scheme is treated extensively in SGA 3, in relation to the classification of tori on an arbitrary base.

In the case $n = 1$, one knows that one can recover (up to equivalence) the 1-groupoid C_1 from the 1-category $\underline{\mathrm{Hom}}(C_1, \mathrm{Set})$ of the functors into $\mathrm{Set} = 0-\mathrm{Cat}$ (i.e. the "local systems" on C_1 which is a topos, called "multigaloisian") - like the category of "fibred functors" on the above topos, i.e. the opposite category to the category of points of this topos (which is none other than the *classifying topos* of C_1). To make precise for arbitrary n the way in which the homotopy n-type of a topos X (supposed for simplicity to be locally homotopically trivial in dim $\leq n$) i.e. its fundamental n-groupoid C_n, can be expressed in terms of the n-category of "local $(n-1)$-systems on X" i.e. of the locally constant $(n-1)$-stacks on X, and to elucidate completely the hypothetical statement (B) above, it is necessary to make explicit how an n-groupoid C_n can be recovered, up n-equivalence, from the knowledge of the n-category

$$\underline{C}_n = n - \underline{\mathrm{Hom}}(C_n, (n-1) - \mathrm{Cat})$$

of local $(n-1)$-systems on C_n. One would like to say that C_n is the category of "fibred n-functors" on $\underline{C}_n$, i.e. of n-functors $\underline{C}_n \longrightarrow (n-1) - \mathrm{Cat}$ having certain exactness properties (for $n = 1$, this is the condition of being the inverse image functor for a morphism of topoi, i.e. to commute with arbitrary $\varprojlim$ and with finite $\varinjlim$...). It is this which makes real the fear, expressed in my preceding letter, that one ends by falling upon the notion of n-topos and of morphisms of these! $\underline{C}_n$ will be an n-topos, (called the "classifying n-topos" of the n-groupoid C_n), $(n-1) - \mathrm{Cat}$ will be the n-topos of points, and C_n will be interpreted modulo n-equivalence as the n-category of "n-points" of the classifying n-topos $\underline{C}_n$. Brr !

If one hopes to be able to define a good old classifying 1-topos for an n-groupoid C_n, as solution of a universal problem, I can see only how to recover the following universal problem: for every topos T, consider $\underline{\mathrm{Hom}}(\Pi_n(T), C_n)$. This is an n-category, but take from it the truncated 1-category $\tau_1\underline{\mathrm{Hom}}(\Pi_n(T), C_n)$. For variable T, one wants to 2-represent the contravariant 2-functor $\mathrm{Top}^\circ \longrightarrow 1 - \mathrm{Cat}$ by a classifying topos $\mathrm{B} = \mathrm{B}_{C_n}$, and then to find a 2-universal $\Pi_n(\mathrm{B}) \xrightarrow{\varphi} C_n$ in the sense that for all T, the functor

$$\underline{\mathrm{Hom}}_{\mathrm{Top}}(T, \mathrm{B}) \xrightarrow{u \mapsto \varphi \circ \Pi_n(u)} \tau_1 \underline{\mathrm{Hom}}(\Pi_n(T), C_n)$$

is an equivalence. For $n = 1$ one knows that the usual classifying topos of C_1 does the job, but for $n = 2$ already, I doubt that this universal problem has a solution. This is perhaps related to the fact that the "Van Kampen Theorem", which one can express by saying that the 2-functor $T \longrightarrow \Pi_1(T)$ of locally 1-connected topoi to groupoids transforms (up to 1-equivalence) amalgamated sums to amalgamated sums (and more generally commutes with inductive 2-limits), is doubtless no longer true for $\Pi_2(T)$. Thus, if T is a topological space which is the union of two closed sets, T_1 and T_2, it is doubtless not true that giving a locally constant 1-stack on T " is equivalent to" giving a locally constant 1-stack F_i on

T_i ($i = 1,2$) and an equivalence between the restrictions of F_1 and F_2 to $T_1 \cup T_2$ (while the analogous statement in terms of 0-stacks, i.e. for coverings, is evidently correct).

The statement (B) above makes it clear how to give explicitly the cohomology of an n-groupoid C_n. If $C_n = \Pi_n(X)$, and if F is a locally constant $(n-1)$-stack on X, and e^X_{n-1} is the "final" $(n-1)$-stack, one has an $(n-1)$-equivalence of $(n-1)$-categories

$$\Gamma_X(F) = F(X) \simeq \mathrm{Hom}(e^X_{n-1}, F)$$

which shows that the functor Γ_X "integration on X" for locally constant $(n-1)$-stacks, which includes the (non-commutative) locally constant cohomology of X in dim $\leq n-1$, can be interpreted in terms of "local $(n-1)$-systems" on the fundamental groupoid as an $\underline{\mathrm{Hom}}(e^{C_n}_{n-1}, F)$ where now F is interpreted as an n-functor

$$C_n \xrightarrow{F} (n-1)-\mathrm{Cat}$$

and $e^{C_n}_{n-1}$ is the constant n-functor on C_n, with value the final $(n-1)$-category.

To interpret this in cohomology notation, it is necessary for me to add, as "apology" to the preceding letter, the explicit interpretation of the non-commutative cohomology on a topos X, in terms of integration of n-stacks on X. If F is a strict Picard n-stack on X, then it is defined by a complex L° on X

$$0 \longrightarrow L^0 \longrightarrow L^1 \longrightarrow L^2 \longrightarrow \ldots \longrightarrow L^n \longrightarrow 0$$

concentrated in degrees $0 \leq i \leq n$ (defined uniquely up to isomorphism in the derived category of $\mathrm{Ab}(X)$). That said, the $\mathrm{H}^i(X, L')$ (hypercohomology) for $0 \leq i \leq n$ can be interpreted as $\mathrm{H}^i(X, L') = \pi_{n-i}\Gamma_X(F)$. If one is interested in all the H^i (not just for $i \leq n$) one must, for all $N \geq n$, regard L° as a complex concentrated in degrees $0 \leq i \leq N$ by prolongation of L° by 0 to the right). The corresponding strict Picard n-stack is no longer F but $\underline{C}^{N-n}F$, where $\underline{C}$ is the "classfying space" functor, interpreted on strict Picard n-categories as the operation consisting of "translating" the i-objects to $(i+1)$-objects, and adjoining a unique 0-object; this extends one hopes, in "an obvious way", to n-stacks, so as to commute with the operation of taking the inverse image of an n-stack. One has then for $i \leq N$

$$\mathrm{H}^i(X, L') = \pi_{N-i}\Gamma_X(C^{N-n}F) \quad i \leq N.$$

Given this, it is necessary to put, for all strict Picard n-stacks F on X,

$$\mathrm{H}^i(X,F) = \pi_{N-i}\Gamma_X(C^{N-n}F) \quad \text{if} \quad N \geq i,n$$

which does nor depend on the choice of integer $N \geq Sup(i,n)$ [**N.B.** One has a canonical morphism of $(n-1)$-groupoids,

$$C(\Gamma_X F) \longrightarrow \Gamma_X(CF),$$

as the obvious constructions in terms of cochains show, and one sees in the same way that this induces isomorphisms on π_i for $1 \leq i \leq n+1$.]

N.B. One sees by the way that for F and n-stack of groupoids on X, if one restricts to defining the $\mathrm{H}^i(X,F)$ for $0 \leq i \leq n$, one has no need of a Picard structure on F, as it is sufficient to put

$$\mathrm{H}^i(X,F) = \pi_{n-i}(\Gamma_X(F)) \quad 0 \leq i \leq n.$$

If on the other hand F is an n-Gr-stack (i.e. F has the structure of a composition law $F \times F \longrightarrow F$ with the usual formal properties of a group) the "classifying $(n+1)$-stack" is defined, and one can define $\mathrm{H}^i(X,F)$ for $i \leq n+1$ by

$$\mathrm{H}^i(X,F) = \pi_{n+1-i}(\Gamma_X(CF))$$

in particular

$$\mathrm{H}^{n+1}(X,F) = \pi_0(\Gamma_X(CF)) = \text{ equivalence classes of sections } \underline{C}F.$$

But one can form $C\underline{C}F = \underline{C}^2F$ and define $\mathrm{H}^{n+2}(X,F)$, it seems *only* if $\underline{C}F$ is itself a Gr-$(n+1)$-stack, which is without doubt the case only if F is a strict Picard n-stack...

Let us now come to the case where F is a *locally constant* n-stack on X, and so is defined by an $(n+1)$-functor

$$C_{n+1} \xrightarrow{F} \text{strict Picard } n-\text{Cat}.$$

Then, putting for $0 \leq i \leq n$

$$\mathrm{H}^i(C_{n+1},F) = \pi_{n-1}(\underline{\mathrm{Hom}}(e_n^{C_{n+1}},F)),$$

"one knows it fails", as one has a canonical isomorphism

$$\mathrm{H}^i(C_{n+1},F) \simeq \mathrm{H}^i(X,F),$$

valid in effect without Picard structure on F... It is thus necessary for all i and for every ∞-groupoid C and every $(n+1)$-functor

$$C \xrightarrow{F} \text{strict Picard } n-\mathrm{Cat},$$

to define

$$\mathrm{H}^i(C,F) = \pi_{N-i}\underline{\mathrm{Hom}}(e_N^C, C^{N-n}F)$$

where one chooses $N \geq Sup(i,n)$. If F has only a Gr-structure (not necessarily Picard) one can define the $\mathrm{H}^i(C,F)$ for $i \leq n+1$ by

$$\mathrm{H}^i(C,F) = \pi_{n+1-i}\underline{\mathrm{Hom}}(e_{n+1}^C, CF).$$

In the case $C = C_{n+1} = \Pi_{n+1}(X)$, it must still be true (by virtue of (A) above), that this set is canonically isomorphic to $\mathrm{H}^{n+1}(X,F) = \pi_0\Gamma_X(CF)$ (this is true and very easy for $n=0$). Can one describe the arrow between the two sides of

$$\mathrm{H}^{n+1}(X,F) \simeq \mathrm{H}^{n+1}(\Pi_{n+1}X,F) \quad ?$$

If one wishes to make (A) and (B) explicit again, in terms of the yoga (C), one comes to the following situation:

One has an $(n+1)$-functor between $(n+1)$-groupoids

$$f_{n+1} : C_{n+1} \longrightarrow D_{n+1}$$

which induces by truncation an n-functor

$$f_n : C_n \longrightarrow D_n$$

One must than have:

(A') f_n is an n-equivalence if and only if the n-functor

$$f_n^* : \underline{\mathrm{Hom}}(D_n,(n-1)-\mathrm{Cat}) \longrightarrow \underline{\mathrm{Hom}}(C_n,(n-1)-\mathrm{Cat})$$

which sends the local $(n-1)$-systems on D_n (or, equally, on D_{n+1}) to the local $(n-1)$-systems on C_n, is an n-equivalence.

(B') f_n is an n-equivalence if and only if for every local n-system F on D_{n+1},

$$F : D_{n+1} \longrightarrow n - \mathrm{Cat},$$

the n-functor induced by f_{n+1}

$$\underbrace{\underline{\mathrm{Hom}}(e_n^{D_{n+1}}, F)}_{\Gamma_{D_{n+1}(F)}} \longrightarrow \underbrace{\underline{\mathrm{Hom}}(e_n^{D_{n+1}}, f_{n+1}^* F)}_{\Gamma_{C_{n+1}(F)}}$$

is an n-equivalence.

The construction of the cohomology of a topos in terms of integration of stacks makes no appeal at all to complexes of abelian sheaves and still less to the technique of injective resolutions. One has the impression that in this spirit, *via* the definition (which remains to be made explicit!) of n-stacks, it is all related above all to the "Cechist" calculations in terms of hypercoverings. Now these last are written with the help of a small dose of semi-simplicial algebra. I do not know if a theory of stacks and of operations on them can be written *without* ever using semi-simplicial algebra. If yes, there would be essentially three distinct approaches for constructing the cohomology of a topos:

a) viewpoint of complexes of sheaves, injective resolutions, derived categories (*commutative homological algebra*)

b) viewpoint Cechist or semi-simplicial (*homotopical algebra*)

c) viewpoint of n-stacks (categorical algebra, or *non-commutative homological algebra*).

In (a) one "resolves" the coefficients, in (b) one resolves the base space (or topos), and in (c) it appears one resolves neither the one nor the other.

Very cordially,

Alexandre