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A. Grothendieck's Early Work (1950–1960)

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The exceptional mathematical talent of Alexander Grothendieck was apparent very soon. He had been a student at the University of Montpellier which, at that time, was among the most backward of French universities in the teaching of mathematics; it is not there that he could have been made aware of the main problems on which contemporary mathematicians were working. His first contact with them occurred in 1948–1949, in the first of the famous Seminars initiated by H. Cartan; the themes chosen for that year were simplicial algebraic topology and the basic notions of sheaf theory (theories taught nowhere else in France at that time). But – surprisingly enough when one thinks of his subsequent career – Grothendieck was not particularly attracted by these subjects, whereas, from what he had heard of functional analysis, he was anxious to learn more in that field. On the advice of Cartan, he went to Nancy in October 1949. At that time, Delsarte, Godement, L. Schwartz and myself had organized a Seminar there on topological vector spaces, a theory on which we all were then working (from different angles).

Banach spaces and their duality were already almost classical around 1950, but the theory of locally convex spaces was barely beginning; only special types of these spaces had been investigated, such as the Köthe spaces of sequences and the spaces of distributions introduced by Sobolev and Schwartz. A general theory of duality for locally convex spaces had to be worked out; Schwartz and I had started its study for Fréchet spaces and their direct limits, but we had met a series of problems which we could not solve. We therefore proposed them to Grothendieck, and the result turned out to exceed our most sanguine expectations. In less than a year, he had solved all our problems by very ingenious new constructions; then, with the techniques he had developed, he started to work on many other questions in functional analysis. When, in 1953, it was time to grant him a doctor's degree, it was necessary to choose from among six papers he had written, any one of which was at the level of a good dissertation. Of course, his fundamental paper on tensor products and nuclear spaces is the one that was chosen and it rapidly became a landmark in functional analysis.

In that long paper, he entered an entirely unexplored domain, the study of 'reasonable' topologies one could define on the tensor product of two locally convex spaces; the case of Banach spaces was the only one which had been investigated. In the deep and original study he made of the general case, Grothendieck already exhibits what will become his style. Although at that time the concept of a category was not yet widely spread, the spirit of that notion pervades the paper, with its persistent search for 'natural' definitions and 'functorial' properties. But, coexistent with these general theorems, at every turn one finds a clever counterexample to test their limits of applicability. The most remarkable discovery in that thesis was that of nuclear spaces, obtained by comparing two possible topologies on tensor products. Although unsuspected until then, they turned out to be the class of spaces that are closest to the finite dimensional ones in their pleasant features. Grothendieck showed that the beautiful results known for spaces of distributions (in particular, Schwartz's 'kernel theorem') were just consequences of the fact that these spaces are nuclear. Since then, nuclear spaces have found many other applications, in particular in probability theory.

Another remarkable novelty in Grothendieck's thesis is the study of continuous linear maps between locally convex spaces that can be *factorized* through suitable $L^1(\mu)$ spaces. Grothendieck later wrote a paper on the 'metric' theory of tensor products, in which these factorizable maps are deeply investigated. That paper has become a classic in the geometry of Banach spaces.

Thus, in less than three years, concepts and results were created whose impact, in my opinion, can only be matched by the work of Banach himself.

But, after giving a course on locally convex spaces at São Paulo in 1953, Grothendieck began to turn to other fields, namely homological algebra, sheaf theory and their applications to algebraic and analytic geometry, which, in the years 1950–1960, were developing at an unprecedented rate. He was helped in his endeavors by Serre, with whom he had exchanged many letters since 1954. But he progressed in these new directions as rapidly as he had done in functional analysis; whereas in 1954 he still acknowledged that he did not feel quite at home with spectral sequences, in 1956 even Serre was astonished by his virtuosity in their manipulation.

In 1955, he was invited to stay for a semester at the University of Kansas, where he chiefly worked on homological algebra and on holomorphic vector bundles. He obtained their complete classification when the base space is the complex projective line, the first step in a theory that is still very active today. Having in mind applications of homological agebra to algebraic geometry, he developed ideas (unknown to him at first) of Buchsbaum and Mac Lane on Abelian categories, extending the work of Cartan–Eilenberg on modules in their well-known book (not yet in print at that time). As usual, he went straight to the heart of the matter by giving a system of axioms for Abelian categories. His most important result was the proof that sheaves of modules on an arbitrary topological space are the objects of an Abelian category with enough injectives. This enabled him to define cohomology with values in a sheaf of modules, with no additional restriction on the sheaf or on the topological space. For many years, this paper remained a classic for the specialists in homological algebra (who called it 'the Tōhoku', after the name of the journal in which it was published).

After his return to France in 1956, Grothendieck took an active part in the work of the Paris mathematical school and he more and more devoted his efforts to algebraic geometry, and assiduously studied the papers of Serre on algebraic varieties over a field k of positive characteristic, which had just been published.

The main trend of his work is the one that will lead him from 'absolute' properties of algebraic varieties to the corresponding 'relative' theorems, the properties of morphisms. Oddly enough, whilst the notion of morphism had been central in most categories – long before the general concept of category had been formulated – in classical algebraic geometry one was chiefly dealing with rational mappings $X \dashrightarrow Y$, which, in general, are not defined everywhere in X. Genuine morphisms, defined everywhere, did not attract much attention, in contrast with what was being done in contemporary papers on analytic varieties (K. Stein, Grauert and Remmert).

In his first result in algebraic geometry, Grothendieck focused his research on the notion of 'proper morphism', the 'relative' counterpart of A. Weil's 'complete variety'. He used it to give a 'relative' version of Serre's theorem proving that, for a complete variety X and a coherent $\mathcal{O}_X$-module $\mathcal{F}$, the cohomology groups $H^j(X; \mathcal{F})$ are finite dimensional vector spaces over the base field k when k is algebraically closed. Grothendieck proved in 1956 that, for a proper morphism $f: X \to Y$, the 'higher images' $R^q f_*(\mathcal{F})$ of a coherent $\mathcal{O}_X$-module $\mathcal{F}$ are coherent $\mathcal{O}_Y$-modules.

It is in the same direction that, a little later, he accomplished his first breakthrough in algebraic geometry, the 'relative' version of the Riemann–Roch–Hirzebruch theorem.

In 1953, building on the work of Cartan–Serre and Kodaira–Spencer on sheaf cohomology on complex varieties, Hirzebruch had proved a general theorem for a complex smooth projective algebraic variety M of dimension m: for a holomorphic complex vector bundle E with base space M, one has, in sheaf cohomology, the general formula

$$\dim H^0(\mathcal{F}) - \dim H^1(\mathcal{F}) + \cdots + (-1)^m \dim H^m(\mathcal{F}) = \kappa_{2m}(\mathrm{ch}(E) \cup \mathrm{td}_M)\,; \qquad (1)$$

$\mathcal{F}$ is the sheaf of germs of holomorphic sections of E over M, $\mathrm{ch}(E)$ the Chern character of E, td_M the sum of the Todd polynomials $T_k(c_1, c_2, \ldots, c_k)$ in the Chern classes c_j of the tangent bundle of M; the right-hand term in (1) is the value $\langle u, [M] \rangle$ at the fundamental class $[M]$ of M, of the sum u of all terms of $\mathrm{ch}(E) \cup \mathrm{td}_M$ belonging to $H^{2m}(M; \mathbb{Q})$. When M is a curve ($m = 1$) and E the line bundle associated with a divisor on M, formula (1) implies the classical Riemann–Roch theorem.

To understand what a 'relative' version of formula (1) might be, suppose, for simplicity, that X, Y are two smooth projective complex varieties with $\dim X =$

m, $\dim Y = n$, and let $f: X \to Y$ be a morphism. The initial goal would be to establish, if possible, a relation between $\mathrm{ch}_X(E) \cup \mathrm{td}_X$ and $\mathrm{ch}_Y(E') \cup \mathrm{td}_Y$, where E' should be a holomorphic vector bundle on Y, depending on f; for $Y = pt.$ $(n = 0)$, one should recover (1). But many hurdles had to be overcome before this idea could be transformed into a genuine mathematical statement.

The first step is that for $z \in H^{\cdot}(X; Q)$, $\kappa_{2m}(z)$ should be replaced by $f_*(z)$, where $f_*: H^{\cdot}(X; \mathbb{Q}) \to H^{\cdot}(Y; \mathbb{Q})$ is a ring homomorphism. Since $H^{\cdot}$ is a contravariant functor, this f_* cannot be a 'natural' map but, using Poincaré duality, it is possible to define homomorphisms

$$f_{*p}: H^p(X; \mathbb{Q}) \to H^{p+d}(Y; \mathbb{Q})$$

with $d = 2(n - m)$; for $n = 0$ one has $f_{*p}(z) = 0$ for $p < 2m$, and $f_{*,2m}(z) = \kappa_{2m}(z)$ for $p = 2m$.

The next difficulty is to have a 'relative' version of the cohomology groups $H^j(\mathscr{F})$. The natural candidates are the Leray 'higher direct images' $R^j f_*(\mathscr{F})$; but if one wanted to stick to the Hirzebruch formula, dealing with vector bundles, $\mathscr{F}$ should be a locally free sheaf. However, the $R^j f_*(\mathscr{F})$ are not locally free in general when $\mathscr{F}$ is locally free; but if $\mathscr{F}$ is coherent, so are the $R^j f_*(\mathscr{F})$, and the 'relative' version of (1) should thus deal with general coherent sheaves $\mathscr{F}$.

But what could replace the left-hand side of (1) when the $H^j(\mathscr{F})$ are replaced by sheaves? It is here that comes the essential new idea, what is now called the *Grothendieck group* $K(Y)$. Let $C(Y)$ be the set of isomorphism classes of coherent $\mathscr{O}_Y$-modules, and consider the $\mathbb{Z}$-module $\mathbb{Z}^{(C(Y))}$ of their formal linear combinations; write $[\mathscr{F}]$ for the class in $C(Y)$ of a coherent $\mathscr{O}_Y$-module $\mathscr{F}$, and $e_{[\mathscr{F}]}$ the corresponding basis element in the free $\mathbb{Z}$-module $\mathbb{Z}^{(C(Y))}$. Then

$$K(Y) = \mathbb{Z}^{(C(Y))}/S(Y) \tag{2}$$

where $S(Y)$ is the $\mathbb{Z}$-submodule generated by all elements

$$e_{[\mathscr{F}]} - e_{[\mathscr{F}']} - e_{[\mathscr{F}'']}$$

such that there is an exact sequence

$$0 \to \mathscr{F}' \to \mathscr{F} \to \mathscr{F}'' \to 0 \tag{3}$$

of coherent $\mathscr{O}_Y$-modules. This is a typical 'universal' object: any *mapping* $\varphi: C(Y) \to G$ into a commutative group G, such that

$$\varphi([\mathscr{F}]) = \varphi([\mathscr{F}']) + \varphi([\mathscr{F}'']) \tag{4}$$

for all exact sequences (3), is uniquely factorized into

$$\varphi: C(Y) \xrightarrow{e} \mathbb{Z}^{(C(Y))} \xrightarrow{\gamma_Y} K(Y) \xrightarrow{\psi} G\,, \tag{5}$$

where $e: \mathscr{F} \mapsto e_{[\mathscr{F}]}$, γ_Y is the natural homomorphism and ψ is a *homomorphism* of groups.

Once the group $K(Y)$ is defined, the alternate sum

$$\sum_q (-1)^q \gamma_Y(e([R^q f_*(\mathscr{F})])) \tag{6}$$

is meaningful as an element of $K(Y)$, and only depends on the class $[\mathscr{F}]$ in $C(X)$; if one denotes it by $\varphi([\mathscr{F}])$, then relation (4) is a consequence of the cohomology exact sequence

$$\cdots \to R^q f_*(\mathscr{F}') \to R^q f_*(\mathscr{F}) \to R^q f_*(\mathscr{F}'') \to R^{q+1} f_*(\mathscr{F}'') \to \cdots$$

which has only a finite number of nonvanishing terms. The element (6) can thus be written

$$f_!(\gamma_X([\mathscr{F}])),$$

where $f_!$: $K(X) \to K(Y)$ is a homomorphism of groups.

One still has to define a Chern character, i.e. a group homomorphism

$$\text{ch}: K(X) \to H^{\cdot}(X; \mathbb{Q}).$$

Grothendieck uses the fact that any coherent $\mathscr{O}_X$-module $\mathscr{F}$ has a *finite* resolution

$$0 \leftarrow \mathscr{F} \leftarrow \mathscr{L}_0 \leftarrow \mathscr{L}_1 \leftarrow \cdots \leftarrow \mathscr{L}_r \leftarrow 0, \tag{7}$$

where the $\mathscr{L}_j$ are locally free sheaves, corresponding to vector bundles over X; the total Chern class $\text{c}(\mathscr{L}_j)$ is therefore defined. The total Chern class $\text{c}(\mathscr{F})$ is then defined as

$$\text{c}(\mathscr{F}) = \text{c}(\mathscr{L}_0)\text{c}(\mathscr{L}_1)^{-1}\text{c}(\mathscr{L}_2)\ldots\text{c}(\mathscr{L}_r)^{(-1)^r}$$

in the ring $H^{\cdot}(X; \mathbb{Q})$; it is independent of the choice of the resolution (7) of $\mathscr{F}$. Then $\text{ch}(\mathscr{F})$ is deduced from $\text{c}(\mathscr{F})$ in the usual way.

With these definitions, the Grothendieck version of the Riemann–Roch theorem is the equality

$$\text{ch}_Y(f_!(x) \cup \text{td}_Y) = f_*(\text{ch}_X(x) \cup \text{td}_X) \tag{8}$$

in the ring $H^{\cdot}(Y; \mathbb{Q})$, for every element $x \in K(X)$.

Grothendieck's proof of (8) is a purely algebraic one: it relies on a new method (later used by other mathematicians), the splitting $f = g \circ h$ of a morphism f: $X \to Y$, where h: $X \to X \times Y$ is the natural injection mapping X onto the graph of f, and g is the second projection $X \times Y \to Y$. One first proves that $f_! = g_! \circ h_!$, and thus reduces the proof of (8) to the proofs of the same relation for g and h. For g the arguments are fairly simple, but they are more intricate for h. The main purpose of Grothendieck was to obtain a proof which could be extended to complete varieties over an algebraically closed field of positive characteristic. To do that he had to replace the cohomology ring $H^{\cdot}(X; \mathbb{Q})$ applicable to complex varieties, by the ring $A(X) \otimes \mathbb{Q}$, where $A(X)$ is the Chow ring of classes of cycles, and the Chern classes have to be defined with values in that ring.

I have gone into some detail concerning this paper, due to its importance as the starting point of K-theory, which has spread to so many parts of mathematics; and also to show the originality and fruitfulness of Grothendieck's imagination. He did not publish his proof of (8), leaving that task to A. Borel and Serre, who animated a Princeton Seminar, using Grothendieck's notes. This was the first example of what became a pattern in his career; having to think about so many ideas that sprung into his mind, he often left to colleagues or students the details of their consequences.

He had barely finished proving his version of the Riemann–Roch theorem when he embarked on a much more ambitious undertaking. In his address to the Edinburgh International Congress in 1958, he disclosed, for the first time, his ultimate objective, which remained central in his work during the next 10 years: to define, for algebraic varieties over fields of *arbitrary* characteristic, cohomology groups with coefficients in a field of *characteristic* 0, with the properties staked out by A. Weil in order to prove his famous conjectures. Without yet unveiling any details, Grothendieck conceived that to reach that goal, he had to conduct a reorganization of algebraic geometry along new lines – similar to the one made by Weil himself for classical algebraic geometry in the light of the proof of his conjectures for curves. Grothendieck was to carry on this gigantic task through the thousands of pages of his papers and Seminars.

Ever since Weil's definition of 'abstract' varieties, not necessarily embeddable in a projective space, the various definitions proposed for them all started with 'affine varieties', which afterwards were 'glued' together. In his famous paper FAC, Serre had observed that an 'affine variety' M, initially conceived as an 'algebraic set' contained in a vector space k^n (for an algebraically closed field k) is in canonical correspondence with a reduced, finitely generated k-algebra, consisting of the 'regular' functions on M. Conversely, such a k-algebra A defines a set M, the set of k-homomorphisms $A \to k$ (or, equivalently, the set of maximal ideals of A); then A defines a topology on M, by taking as a basis of open sets the sets $D(f) = \{x \in M \mid f(x) \neq 0\}$ for all $f \in A$; and finally a sheaf $\mathcal{O}_M$ of local rings, by the condition that the ring of sections $\Gamma(D(f), \mathcal{O}_M)$ be the localized ring A_f for every $f \in A$.

Chevalley animated, jointly with H. Cartan, a Seminar on algebraic geometry in 1955–1956. He introduced objects which he called *schemes*, by gluing together 'affine schemes'; for certain properties, the latter were more general than Serre's affine varieties, but more restrictive in other features. The starting point was again a k-algebra A, but the field k was not necessarily algebraically closed and A was restricted to be an integral domain. There were no sheaves in Chevalley's theory; the 'affine scheme' defined by A is the set of all localized rings $A_\mathfrak{p}$ for *prime* ideals $\mathfrak{p}$, not only maximal ideals. The 'Zariski topology' on M is defined by taking for all closed sets

$$E(\mathfrak{a}) = \{A_\mathfrak{p} \mid \mathfrak{p} \supset \mathfrak{a}\}$$

for all ideals $\mathfrak{a}$ of A. A little later, Nagata extended Chevalley's definitions by replacing the field k by a Dedekind ring in the definition.

Grothendieck had followed the Cartan–Chevalley Seminar, and he knew Nagata's paper. The obvious relations between all these definitions and commutative algebra led him to throw away the various restrictions imposed by Serre, Chevalley, and Nagata, in order to reach the most useful conception of what algebraic geometry should be. He therefore called *affine scheme* the 'prime spectrum' $\mathrm{Spec}(A) = X$, the set of all prime ideals of an *arbitrary* commutative ring A with unit element. A topology is defined on X by taking as a basis of open sets on X the sets

$$D(f) = \{\mathfrak{p} \mid f \not\in \mathfrak{p}\}$$

for all $f \in A$; finally, a sheaf $\mathcal{O}_X$ of local rings on A is defined by the condition that $\Gamma(D(f), \mathcal{O}_X) = A_f$ for all $f \in A$; its stalk at the point $\mathfrak{p}$ is the local ring $A_\mathfrak{p}$. A general *scheme* $(X, \mathcal{O}_X)$ is a ringed space having a basis of open sets (U_α) such that $(U_\alpha, \mathcal{O}_X \mid U_\alpha)$ is an affine scheme; the open sets having this property are called the affine open sets.

Schemes are the objects of a category; the morphisms of that category are defined as morphisms of spaces equipped with sheaves of local rings. Affine schemes constitute a full subcategory, which is equivalent to the dual category of all commutative rings with unit element.

In his 1958 address, Grothendieck had already sketched these definitions, and stressed the two main directions in which he wanted to develop the theory. The first one was to emphasize morphisms more than schemes: once a 'base scheme' S has been fixed (it can be any scheme), the genuine objects of study are the pairs (X, f) consisting of a scheme X and a morphism $f: X \to S$. They are called *S-schemes*, and they are the objects of a category; the morphisms of that category (called the S-morphisms), written $(X, f) \to (Y, g)$, are the morphisms of schemes $h: X \to Y$ such that $g \circ h = f$; S thus plays the part of the 'base field' in classical algebraic geometry.

The other main direction was the extension to schemes of the sheaf cohomology techniques inaugurated by Serre; Grothendieck extended them from coherent sheaves to quasi-coherent ones; on an affine scheme $\mathrm{Spec}(A)$, they correspond to *arbitrary* A-modules.

From 1960 to 1970, Grothendieck was a permanent member of the newly founded 'Institut des Hautes Etudes Scientifiques' (IHES), in which he animated a Seminar on algebraic geometry; it soon attracted many students, to whom he very generously suggested research themes connected with the theory of schemes, and provided useful guidance.

During that time, he organized the publication of his results on two levels. The basic notions of the theory of schemes were given a systematic and detailed exposition in his *Eléments de Géométrie algébrique* (EGA) published in successive instalments in the 'Publications mathématiques de l'IHES'. At the same time, the more advanced parts of the theory were the material of rather succinct talks at the Bourbaki Seminar and, in more elaborate form, in Grothendieck's own Seminar at IHES, where he was helped by colleagues and students in their publication.

Grothendieck's influence very soon spread and became enormous. Already in 1962, in an address to the International Congress at Stockholm, Serre could say that the theory of schemes was the best frame for the development of algebraic geometry. This is completely obvious nowadays; every student who wishes to learn algebraic geometry must become familiar with the theory of schemes; several good textbooks make it easier to assimilate than the ponderous EGA.

I leave to more competent specialists the description and evaluation of the more than 6000 pages of the Seminars in which Grothendieck, with some help from his colleagues, built up the powerful new tools that he introduced in algebraic geometry: descent criteria, Hilbert and Picard schemes, derived categories, residues and duality, group schemes and, above all, the 'Grothendieck topologies' that, after much toil, finally provided the ultimate goal towards which he had been striving, the cohomologies with coefficients in a field of characteristic 0. It is well known that the properties of these cohomologies enabled Deligne to prove the Weil conjectures, and Faltings the Mordell conjecture. But it is appropriate to stress that these beautiful results should not be dissociated from the tremendous surge of new concepts and theorems which have made algebraic geometry one of the most active and fruitful parts of mathematics of the last 40 years, including the spawning ground of *K*-theory.