Gr-CATEGORIES[1]
Summary

The purpose of these notes is to study the Gr-categories and give some applications of them. Below is a brief description of the organisation of the work.

Chapter I gives some definitions and results, which are used continually in the sequel, on $\otimes$-categories one can find in [2], [6], [11], [14], [15], the terminology employed in this chapter being of Neantro Saavedra Rivano [14]. A *$\otimes$-category* is a category $\mathscr{C}$ together with a *law* $\otimes$, i.e. a covariant bifunctor

$$\otimes : \mathscr{C} \times \mathscr{C} \longrightarrow \mathscr{C}$$

$$(X, Y) \mapsto X \otimes Y$$

An *associativity constraint* for a $\otimes$-category $\mathscr{C}$ is an isomorphism of bifunctors

$$a_{X,Y,Z} : X \otimes (Y \otimes Z) \xrightarrow{\sim} (X \otimes Y) \otimes Z, \quad X, Y, Z \in Ob(\mathscr{C})$$

satisfying the *pentagon axiom*, i.e. all the pentagonal diagrams

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are commutative. A $\otimes$-category together with an associativity constraint is called a *$\otimes$-associativity category*.

A *commutativity constraint* for a $\otimes$-category $\mathscr{C}$ is an isomorphism of bifunctors

$$c_{X,Y} : X \otimes Y \xrightarrow{\sim} Y \otimes X, \quad X, Y \in Ob(\mathscr{C})$$

[1] This text had been transcribed by Mateo Carmona
https://agrothendieck.github.io/

verifying the relation

$$c_{Y,X} \circ c_{X,Y} = \mathrm{Id}_{X \otimes Y}$$

The commutativity constraint c is said to be *strict* if $c_{X,X} = \mathrm{Id}_{X\otimes}$ for all $X \in Ob(\mathscr{C})$. A ⊗-category together with a commutativity constraint is a ⊗-*commutative category*. A ⊗-commutative category is *strict* if its commutativity constraint is strict.

An *unity constraint* for a ⊗-category $\mathscr{C}$ is a triple $(\underline{1}, g, d)$ where $\underline{1}$ is an object of $\mathscr{C}$, g and d natural isomorphisms

$$g_X : X \xrightarrow{\sim} \underline{1} \otimes X, \quad d_X : X \xrightarrow{\sim} X \otimes \underline{1}, \quad X \in Ob(\mathscr{C})$$

such that $g_{\underline{1}} = d_{\underline{1}}$. A ⊗-category together with an unity constraint is a ⊗-*unifer category*.

A ⊗-category $\mathscr{C}$ together with an associativity constraint a and a commutaivity constraint c is a ⊗-*AC category* if the *hexagonal axiom* is fulfilled, i.e. all the hexagonal diagram commutes

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A ⊗-category $\mathscr{C}$ together with a associativity constraint a and an unity contraint $(\underline{1}, g, d)$ is a ⊗-*AU category* if all the following triangles commute

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A ⊗-*ACU category* is a ⊗-AC and AU category. An object X of a ⊗-ACU category $\mathscr{C}$ is *invertible* if there are two objects $X', X'' \in Ob(\mathscr{C})$ such that $X' \otimes X \simeq X \otimes X'' \simeq \underline{1}$.

A ⊗-*functor* from a ⊗-category $\mathscr{C}$ to a ⊗-category $\mathscr{C}'$ is a pair $(F, \breve{F})$ where F is a functor $\mathscr{C} \longrightarrow cC'$ and $\breve{F}$ an isomorphism of bifunctors

$$\breve{F}_{X,Y} : FX \otimes FY \longrightarrow F(X \otimes Y) \quad X, Y \in Ob(\mathscr{C})$$

A ⊗-functor $(F, \breve{F})$ from a ⊗-associative category $\mathscr{C}$ to a ⊗-associative category $\mathscr{C}'$ is *associative* if the following diagram commutes:

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where a is the associativity constraint of $\mathscr{C}$ and a' of $\mathscr{C}'$.

A ⊗-functor $(F, \breve{F})$ from a ⊗-commutative category $\mathscr{C}$ to a ⊗-commutative category $\mathscr{C}'$ is *commutative* if the following diagram commutes :

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c and c' being the commutativity constraints of $\mathscr{C}$ and $\mathscr{C}'$ respectively.

A $\otimes$-functor $(F, \breve{F})$ from a $\otimes$-category $\mathscr{C}$ with an unity constraint $(\underline{1}, g, d)$ to a $\otimes$-category $\mathscr{C}'$ with an unity constraint $(\underline{1}', g', d')$ is a$\otimes$-*unifer functor* if there exists an isomorphism $\hat{F} : \underline{1}' \xrightarrow{\sim} F\underline{1}$ such that the following diagrams commute:

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It follows from the definition that the isomorphism $\hat{F} : \underline{1}' \xrightarrow{\sim} F\underline{1}$, it it exists, is unique.

A $\otimes$-*AC functor* is an $\otimes$-associative and commutative functor.

A $\otimes$-*ACU functor* is a $\otimes$-associative, commutative and unifer functor.

Let $(F, \breve{F})$ and $(G, \breve{G})$ be $\otimes$-functors from a $\otimes$-category $\mathscr{C}$ to a $\otimes$-category $\mathscr{C}'$. A $\otimes$-*morphism* from the $\otimes$-functor $(F, \breve{F})$ to the $\otimes$-functor $(G, \breve{G})$ is a morphism of functors $\lambda : F \longrightarrow G$ such that the following diagram commutes

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Chapter II is a study of Gr-categories and Pic-categories. A Gr-*category* is a $\otimes$-AU category, the objects of which are all invertible, and the base category a groupoid (i.e. all arrows are isomorphisms). Thus a Gr-category is like a group. We obtain from this definition that if $\mathscr{P}$ is a Gr-category, the set $\pi_0(\mathscr{P})$ of the classes up to isomorphism of objects of $\mathscr{P}$, together with the operation induced by the law $\otimes$ of $\mathscr{P}$, is a group; the group $\mathrm{Aut}(\underline{1}) = \pi_1(\mathscr{P})$ is a commutative group; and for all $X \in Ob(\mathscr{P})$

$$\gamma_X : u \mapsto u \otimes \mathrm{Id}_X = \mathrm{Aut}(\underline{1}) \xrightarrow{\sim} \mathrm{Aut}(X)$$

$$\delta_X : u \mapsto \mathrm{Id}_X \otimes u = \mathrm{Aut}(\underline{1}) \xrightarrow{\sim} \mathrm{Aut}(X)$$

We attribute thus to a Gr-category $\mathscr{P}$ two groups $\pi_0(\mathscr{P})$ and $\pi_1(\mathscr{P})$ where $\pi_1(\mathscr{P})$ is commutative. Furthermore we can define an action of $\pi_0(\mathscr{P})$ on $\pi_1(\mathscr{P})$ by the formula

$$su = \delta_X^{-1} \gamma_X(u)$$

for $s \in \pi_0(\mathscr{P})$ represents d by X and $u \in \pi_1(\mathscr{P})$. The commutative group $\pi_1(\mathscr{P})$ together with this action is a left $\pi_0(\mathscr{P})$-module.

Let M be a group, N a left M-module. A *preepinglage* of type (M,N) for a Gr-category $\mathscr{P}$ is a pair $\varepsilon = (\varepsilon_0, \varepsilon_1)$ of isomorphisms

$$\varepsilon_0 : M \xrightarrow{\sim} \pi_0(\mathscr{P}), \quad \varepsilon_1 : N \xrightarrow{\sim} \pi_1(\mathscr{P})$$

compatible wit the action of M on N, $\pi_0(\mathscr{P})$ on $\pi_1(\mathscr{P})$. A Gr-category *preeplingled* of type (M,N) is a Gr-category $\mathscr{P}$ together with preepinglage. Finally, an *arrow* of Gr-categories preepingled of type (M,N) $(\mathscr{P},\varepsilon) \longrightarrow (\mathscr{P}',\varepsilon')$ is a $\otimes$-associative functor such that the following triangles commute:

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It follows from this definition that a such arrow is a $\otimes$-equivalence. Thus the set of the equivalence classes of Gr-categories preepingled of type (M,N) is equal to the set of connected components of the category of Gr-categories preepingled of type (M,N).

If we consider the cohomology group $\mathrm{H}^3(M,N)$ of the group M with coefficients N (in the sense of the group cohomology [12]) we obtain a canonical bijection between the set $\mathrm{H}^3(M,N)$ and the set of the equivalence classes of Gr-categories preepingled of type (M,N).

A Pic-*category* is a Gr-category together with a commutativity constraint which is compatible with its associativity constraint, i.e. the hexagon axiom is satisfied. Thus a Pic-category is like a commutative group. We verify immediately that a necessary condition for the existence of a Pic-category structure on a Gr-category is that $\pi_0(\mathscr{P})$ must be commutative and act trivially on $\pi_1(\mathscr{P})$. A Pic-category is *strict* if its commutativity constraint is strict.

Let M, N be abelian groups. A *preepinglage* of type (M,N) for a Pic-category $\mathscr{P}$ is a pair $\varepsilon = (\varepsilon_0, \varepsilon_1)$ of isomorphisms

$$\varepsilon_0 : M \xrightarrow{\sim} \pi_0(\mathscr{P}), \quad \varepsilon_1 : N \xrightarrow{\sim} \pi_1(\mathscr{P})$$

A Pic-category *preepingled* of type (M,N) is a Pic-category together with a preepinglage. We define the *arrow* of such objects in the same way as for Gr-categories.

For next propositions, let us consider two complexes of free abelian groups

$$L_\bullet(M) : L_3(M) \xrightarrow{d_3} L_2(M) \xrightarrow{d_2} L_1(M) \xrightarrow{d_1} L_0(M) \longrightarrow M$$

$$'L_\bullet(M):' L_3(M) \xrightarrow{'d_3 \ '} L_2(M) \xrightarrow{'d_2 \ '} L_1(M) \xrightarrow{'d_1 \ '} L_0(M) \longrightarrow M$$

where

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so that $L_\bullet(M)$ is a truncated resolution of M. One obtains a canonical bijection between the set of the equivalence classes of Pic-categories preepingled of type (M,N) and the set $\mathrm{H}^2(Hom('L_\bullet(M),N))$. The exactitude of the complex $L(M)$ gives us e triviality of the classification of Pic-categories preepingled of type (M,N) which are strict, i.e. all Pic-categories preepingled of type (M,N) which are strict, are equivalent.

Finally chapter III gives us the construction of the solution of two universal problems: *problem of making objects "unity objects"* and *problem of reversing objects*.

Let be a ⊗-AC category, $'$ another ⊗-AC category whose base category is a groupoid, and $(T,\breve{T}):' \longrightarrow$ a ⊗-AC functors. We try to make the objects TA' of , $A' \in Ob(')$, "unity object", i.e. we try to get:

1°) A ⊗-ACU category $\mathscr{P}$

2°) A ⊗-AC functor $(D,\breve{D}):\longrightarrow \mathscr{P}$

3°) A ⊗-isomorphism

$$\lambda:(,\breve{D})\circ(T,\breve{T}) \xrightarrow{\sim} (I_{\mathscr{P}},\breve{I}_{\mathscr{P}})$$

where $(I_{\mathscr{P}},\breve{I}_{\mathscr{P}})$ is the ⊗-constant functors $\underline{1}_{\mathscr{P}}$ from $'$ to $\mathscr{P}$. The triple $(\mathscr{P},(D,\breve{D}),\lambda))$ must be universal for triples $(,(E,\breve{E}),\mu)$ satisfying 1°, 2°, 3°.

For the description of the triple $(\mathscr{P},(D,\breve{D}),\lambda)$, we introduce a quotient category of a ⊗-AC category as follows:

Let be a ⊗-AC category, Y a *multiplicative subset* of (that means a subset of the set of all endomorphisms of such that $Id_X \in Y$ for all $X \in Ob()$ and the tensor product of two arrows of Y belongs to Y). The *⊗-AC category quotient* A^Y of with respect to Y is the solution of the universal problem

$$(K,\breve{K}):\longrightarrow, \quad K(u)=\mathrm{Id} \text{ for all } u \in Y$$

where B is a ⊗-AC category and $(K,\breve{K})$ a ⊗-AC functor.

Now let us give an idea of the construction of the triple $(\mathscr{P},(D,\breve{D},\lambda))$ for $' \neq \varnothing$:

1° $\mathrm{Ob}(\mathscr{P}) = \mathrm{Ob}()$

2° $Hom_{\mathscr{P}}(A,B) = \varphi(A,B)_{/R_{A,B}}, A,B \in \mathrm{Ob}(\mathscr{P})$

$\varphi(A,B)$ being the set of all triples (A',B',u) where $A',B' \in \mathrm{Ob}(')$, $u \in Fl()$, $u : A \otimes TA' \longrightarrow B \otimes TB'$; $R_{A,B}$ the equivalence relation defined in $\varphi(A,B)$ as follows

$$(A'_1, B'_1, u) R_{A,B} (A'_2, B'_2, u)$$

if and only if there are objects C'_1, C'_2 and isomorphisms

$$u' : A'_1 \otimes C'_1 \xrightarrow{\sim} A'_2 \otimes C'_2, \quad v' : B'_1 \otimes C'_1 \xrightarrow{\sim} B'_2 \otimes C'_2$$

of $'$ such that the following diagram commutes in $^{\varphi}$ ⊗-AC quotient category of with respect to the multiplicative subset of generated by the endomorphisms of the form $T(c_{A',A'})$;

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We denote by $[A',B',u]$ the class which has (A',B',u) as representative

3° Composition of arrows in $\mathscr{P}$. Let $[A',B,u] : A \longrightarrow B, [B'',C'',v] : B \longrightarrow C$ be arrows in $\mathscr{P}$. We define

$$[B'',C'',v] \circ [A',B',u] = [A' \otimes B'', B' \otimes C', w] : A \longrightarrow C$$

where w is such that the following diagram commutes:

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4° ⊗-structure on $\mathscr{P}$

$$A \otimes E \text{ (in } \mathscr{P}) = A \otimes E \text{ (in)}$$

$$[A',B',u] \otimes [E',F',v] = [A' \otimes E', B' \otimes F', w]$$

where w is defined by the commutative diagram (1)

5° ACU constraint in $\mathscr{P}$.

$([A',A',a \otimes \mathrm{Id}], [A',A',c \otimes Id], (1_{\mathscr{P}} = TA'_0, g_A = [A'_0 \otimes A', A', t_A], d_A = [A'_0 \otimes A', A', p_A]))$

where A'_0 is a fixed object of $'$, A' an arbitrary object of $'$, g_A and d_A natural isomorphisms

$$g_A : A \longrightarrow 1_{\mathscr{P}} \otimes A, \quad d_A : A \longrightarrow A \otimes 1_{\mathscr{P}}$$

with t_A and p_A defined by the commutativity diagrams (2)

6° $(D, \breve{D})$ is defined by

$$DA = A, \quad D_u = [A', A', u \otimes \mathrm{Id}_{TA'}], \quad \breve{D}_{A,B} = \mathrm{Id}_{A \otimes B}$$

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For the problem of reversing objects, let us consider a ⊗-category $\mathscr{C}$ with a ACU constraint $(a, c, (1, g, d))$ a ⊗-category $\mathscr{C}'$ with a ACU constraint $(a', c', (1', g', d'))$, the base category of which is a groupoid, and a ⊗-ACU functor $(F, \breve{F}) : \mathscr{C}' \longrightarrow \mathscr{C}$. We try to find a ⊗-ACU category $\mathscr{P}$ and a ⊗-ACU functor $(D, \breve{D}) : \mathscr{C} \longrightarrow \mathscr{P}$ having the following properties

1° DFX' is invertible in $\mathscr{P}$ for all $X' \in \mathrm{Ob}(\mathscr{C}')$

2° For all ⊗-ACU functor $(E, \breve{E})$ from $\mathscr{C}$ to a ⊗-ACU category such that EFX' is invertible in for all $X' \in \mathrm{Ob}(\mathscr{C}')$, there exists a ⊗-ACU functor $(E', \breve{E}')$, unique up to ⊗-isomorphism, from $\mathscr{P}$ to such that $(E, \breve{E}) \simeq (E', \breve{E}' \circ (D, \breve{D}))$.

This problem is reduced by the first by putting $' = \mathscr{C}', = \mathscr{C} \times \mathscr{C}'$, $TX' = (FX', X')$ and by remarking that if $\mathscr{C}$, $\mathscr{C}'$, are ⊗-ACU categories, $^{\otimes, ACU}(\mathscr{C},)$ the category of all ⊗-ACU functors from $\mathscr{C}$ to , then there is a canonical equivalence of categories

$$^{\otimes, ACU}(\mathscr{C} \times \mathscr{C}',) \longrightarrow ^{\otimes, ACU}(\mathscr{C},) \times ^{\otimes, ACU}(\mathscr{C}',)$$

The ⊗-ACU category $\mathscr{P}$ thus defined is called the *⊗-category of fractions* of the category $\mathscr{C}$ with respect to $(\mathscr{C}', (F, \breve{F}))$. The ⊗-category of fractions of $\mathscr{C}^{is}$ with respect to $(\mathscr{C}^{is}, (\mathrm{Id}_{\mathscr{C}^{is}}, \mathrm{Id}))$ is a Pic-category which is called the Pic-*envelope* of the category $\mathscr{C}$, and denoted by $\mathrm{Pic}(\mathscr{C})$.

For an application of the Pic-envelope, we take $\mathscr{C} = P(R)$, category of all finitely generated R-modules (R a ring) and $\mathscr{P} = \mathrm{Pic}(P(R))$, then one obtain

$$\pi_0(\mathscr{P}) \simeq K^0(R)$$

$$\pi_1(\mathscr{P}) \simeq K^1(R)$$

where $K^0(R)$is the Grothendieck group and $K^1(R)$ the whitehead group [1].

The use of the ⊗-category of fractions of a ⊗-ACU category gives us the following result:

Let $\mathscr{C}$ be a ⊗-ACU category, Z an arbitrary object of $\mathscr{C}$ different from the unity object $\underline{1}$, S the functor from $\mathscr{C}$ to $\mathscr{C}$ defined by

$$X \mapsto X \otimes Z.$$

The *suspension category* of the ⊗-ACU category $\mathscr{C}$ defined by the object Z is the triple $(\mathscr{P}, i, p)$ which solves the universal problem for triples $(, j, q)$ where is a category, j a functor from $\mathscr{C}$ to , and q an equivalence of categories from to , so that the following diagram commutes

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up to natural isomorphism. In the case where $\mathscr{C}$ is the homotopy category of pointed topological spaces $_*$ together with the smash [] (the smash [] of two spaces X and Y, with the base points x_0 and y_0, is obtained from the product $X \times Y$ by [] the subset [] to a single point which is taken as the base point of []), and the usual ACU constraint; and Z is the 1-sphere S^1 hence S^1 is the suspension functor, we get the well-known definition of the suspension category.

Let $\mathscr{C}'$ be the ⊗-stable subcategory of $\mathscr{C}$ generated by Z and $\mathscr{P}$ the ⊗-category of fractions of $\mathscr{C}$ with respect to $(\mathscr{C}', (F, \mathrm{Id}))$ where $F : \mathscr{C}' \longrightarrow \mathscr{C}$ is the inclusion functor. One obtains a functor $G : \mathscr{P} \longrightarrow \mathscr{P}$ from the suspension category to the ⊗-category of fractions of $\mathscr{P}$. If G is not faithful, that is the case of the homotopy category of pointed topological spaces $_*$ together with the smash ∧ and the 1-sphere S^1; then it is impossible to construct in $\mathscr{P}$ a law ⊗ such that $\mathscr{P}$ together with this law is a ⊗-ACU category, iZ invertible in $\mathscr{P}$, and i embedded in a pair $(i, \breve{i})$ which is a ⊗-ACU functor from $\mathscr{C}$ to $\mathscr{P}$.

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