The purpose of these notes is to study the Gr-categories and give some applications of them. Below is a brief description of the organisation of the work.

Chapter I gives some definitions and results, which are used continually in the sequel, on $\otimes$-categories one can find in [2], [6], [11], [14], [15], the terminology employed in this chapter being of Neantro Saavedra Rivano [14]. A $\otimes$-category is a category $\mathcal{C}$ together with a law $\otimes$, i.e. a covariant bifunctor

$$\otimes : \mathcal{C} \times \mathcal{C} \longrightarrow \mathcal{C}$$

$$(X, Y) \mapsto X \otimes Y$$

An associativity constraint for a $\otimes$-category $\mathcal{C}$ is an isomorphism of bifunctors

$$a_{X,Y,Z} : X \otimes (Y \otimes Z) \xrightarrow{\sim} (X \otimes Y) \otimes Z, \quad X, Y, Z \in Ob(\mathcal{C})$$

satisfying the pentagon axiom, i.e. all the pentagonal diagrams

are commutative. A $\otimes$-category together with an associativity constraint is called a $\otimes$-associativity category.

A commutativity constraint for a $\otimes$-category $\mathcal{C}$ is an isomorphism of bifunctors

$$c_{X,Y} : X \otimes Y \xrightarrow{\sim} Y \otimes X, \quad X, Y \in Ob(\mathcal{C})$$
verifying the relation
\[ c_{Y,X} \circ c_{X,Y} = \text{Id}_{X \otimes Y} \]

The commutativity constraint \( c \) is said to be strict if \( c_{X,X} = \text{Id}_{X \otimes} \) for all \( X \in \text{Ob}(\mathcal{C}) \). A \( \otimes \)-category together with a commutativity constraint is a \( \otimes \)-commutative category. A \( \otimes \)-commutative category is strict if its commutativity constraint is strict.

An unity constraint for a \( \otimes \)-category \( \mathcal{C} \) is a triple \((1, g, d)\) where \( 1 \) is an object of \( \mathcal{C} \), \( g \) and \( d \) natural isomorphisms
\[ g_X : X \cong 1 \otimes X, \quad d_X : X \cong X \otimes 1, \quad X \in \text{Ob}(\mathcal{C}) \]
such that \( g_1 = d_1 \). A \( \otimes \)-category together with an unity constraint is a \( \otimes \)-unifer category.

A \( \otimes \)-category \( \mathcal{C} \) together with an associativity constraint \( a \) and a commutativity constraint \( c \) is a \( \otimes \)-AC category if the hexagonal axiom is fulfilled, i.e. all the hexagonal diagram commutes
\[
\[
\]
A \( \otimes \)-category \( \mathcal{C} \) together with an associativity constraint \( a \) and an unity constraint \((1, g, d)\) is a \( \otimes \)-AU category if all the following triangles commute
\[
\[
\]
A \( \otimes \)-ACU category is a \( \otimes \)-AC and AU category. An object \( X \) of a \( \otimes \)-ACU category \( \mathcal{C} \) is invertible if there are two objects \( X', X'' \in \text{Ob}(\mathcal{C}) \) such that \( X' \otimes X \cong 1 \).

A \( \otimes \)-functor from a \( \otimes \)-category \( \mathcal{C} \) to a \( \otimes \)-category \( \mathcal{C}' \) is a pair \((F, \tilde{F})\) where \( F \) is a functor \( \mathcal{C} \longrightarrow c\mathcal{C}' \) and \( \tilde{F} \) an isomorphism of bifunctors
\[ \tilde{F}_{X,Y} : FX \otimes FY \longrightarrow F(X \otimes Y) \quad X, Y \in \text{Ob}(\mathcal{C}) \]
A \( \otimes \)-functor \((F, \tilde{F})\) from a \( \otimes \)-associative category \( \mathcal{C} \) to a \( \otimes \)-associative category \( \mathcal{C}' \) is associative if the following diagram commutes:
\[
\[
\]
where \( a \) is the associativity constraint of \( \mathcal{C} \) and \( a' \) of \( \mathcal{C}' \).

A \( \otimes \)-functor \((F, \tilde{F})\) from a \( \otimes \)-commutative category \( \mathcal{C} \) to a \( \otimes \)-commutative category \( \mathcal{C}' \) is commutative if the following diagram commutes:
\[c\] and \(c'\) being the commutativity constraints of \(\mathcal{C}\) and \(\mathcal{C}'\) respectively.

A \(\otimes\)-functor \((F, \tilde{F})\) from a \(\otimes\)-category \(\mathcal{C}\) with an unity constraint \((1, g, d)\) to a \(\otimes\)-category \(\mathcal{C}'\) with an unity constraint \((1', g', d')\) is a \(\otimes\)-unifer functor if there exists an isomorphism \(\hat{F} : 1' \sim \rightarrow F_1\) such that the following diagrams commute:

\[
\begin{array}{c}
\end{array}
\]

It follows from the definition that the isomorphism \(\hat{F} : 1' \sim \rightarrow F_1\), if it exists, is unique.

A \(\otimes\)-AC functor is an \(\otimes\)-associative and commutative functor.

A \(\otimes\)-ACU functor is a \(\otimes\)-associative, commutative and unifer functor.

Let \((F, \tilde{F})\) and \((G, \tilde{G})\) be \(\otimes\)-functors from a \(\otimes\)-category \(\mathcal{C}\) to a \(\otimes\)-category \(\mathcal{C}'\). A \(\otimes\)-morphism from the \(\otimes\)-functor \((F, \tilde{F})\) to the \(\otimes\)-functor \((G, \tilde{G})\) is a morphism of functors \(\lambda : F \rightarrow G\) such that the following diagram commutes:

\[
\begin{array}{c}
\end{array}
\]

Chapter II is a study of Gr-categories and Pic-categories. A Gr-category is a \(\otimes\)-AU category, the objects of which are all invertible, and the base category a groupoid (i.e. all arrows are isomorphisms). Thus a Gr-category is like a group. We obtain from this definition that if \(\mathcal{P}\) is a Gr-category, the set \(\pi_0(\mathcal{P})\) of the classes up to isomorphism of objects of \(\mathcal{P}\), together with the operation induced by the law \(\otimes\) of \(\mathcal{P}\), is a group; the group \(\text{Aut}(1) = \pi_1(\mathcal{P})\) is a commutative group; and for all \(X \in \text{Ob}(\mathcal{P})\)

\[
\gamma_X : u \mapsto u \otimes \text{Id}_X = \text{Aut}(1) \sim \rightarrow \text{Aut}(X)
\]

\[
\delta_X : u \mapsto \text{Id}_X \otimes u = \text{Aut}(1) \sim \rightarrow \text{Aut}(X)
\]

We attribute thus to a Gr-category \(\mathcal{P}\) two groups \(\pi_0(\mathcal{P})\) and \(\pi_1(\mathcal{P})\) where \(\pi_1(\mathcal{P})\) is commutative. Furthermore we can define an action of \(\pi_0(\mathcal{P})\) on \(\pi_1(\mathcal{P})\) by the formula

\[
su = \delta_X^{-1} \gamma_X(u)
\]

for \(s \in \pi_0(\mathcal{P})\) represents \(d\) by \(X\) and \(u \in \pi_1(\mathcal{P})\). The commutative group \(\pi_1(\mathcal{P})\) together with this action is a left \(\pi_0(\mathcal{P})\)-module.
Let $M$ be a group, $N$ a left $M$-module. A preplinage of type $(M, N)$ for a Gr-category $\mathcal{P}$ is a pair $\varepsilon = (\varepsilon_0, \varepsilon_1)$ of isomorphisms

$$\varepsilon_0 : M \to \pi_0(\mathcal{P}), \quad \varepsilon_1 : N \to \pi_1(\mathcal{P})$$

compatible with the action of $M$ on $N$, $\pi_0(\mathcal{P})$ on $\pi_1(\mathcal{P})$. A Gr-category preepli-ngled of type $(M, N)$ is a Gr-category $\mathcal{P}$ together with preplinage. Finally, an arrow of Gr-categories preeplngled of type $(M, N)$ $(\mathcal{P}, \varepsilon) \to (\mathcal{P}', \varepsilon')$ is a $\otimes$-associative functor such that the following triangles commute:

It follows from this definition that such an arrow is a $\otimes$-equivalence. Thus the set of the equivalence classes of Gr-categories preeplngled of type $(M, N)$ is equal to the set of connected components of the category of Gr-categories preeplngled of type $(M, N)$.

If we consider the cohomology group $H^3(M, N)$ of the group $M$ with coefficients $N$ (in the sense of the group cohomology [12]) we obtain a canonical bijection between the set $H^3(M, N)$ and the set of the equivalence classes of Gr-categories preeplngled of type $(M, N)$.

A Pic-category is a Gr-category together with a commutativity constraint which is compatible with its associativity constraint, i.e. the hexagon axiom is satisfied. Thus a Pic-category is like a commutative group. We verify immediately that a necessary condition for the existence of a Pic-category structure on a Gr-category is that $\pi_0(\mathcal{P})$ must be commutative and act trivially on $\pi_1(\mathcal{P})$. A Pic-category is strict if its commutativity constraint is strict.

Let $M, N$ be abelian groups. A preplinage of type $(M, N)$ for a Pic-category $\mathcal{P}$ is a pair $\varepsilon = (\varepsilon_0, \varepsilon_1)$ of isomorphisms

$$\varepsilon_0 : M \to \pi_0(\mathcal{P}), \quad \varepsilon_1 : N \to \pi_1(\mathcal{P})$$

A Pic-category preeplngled of type $(M, N)$ is a Pic-category together with a preplinage. We define the arrow of such objects in the same way as for Gr-categories.

For next propositions, let us consider two complexes of free abelian groups

$$L_4(M) : L_3(M) \to L_2(M) \to L_1(M) \to L_0(M) \to M$$
\[
\begin{array}{ccc}
\prime L_3(M) & \xrightarrow{d_3} & \prime L_2(M) \\
\xrightarrow{d_2} & & \xrightarrow{d_1} \\
\prime L_1(M) & \xrightarrow{d_1} & \prime L_0(M) \\
\xrightarrow{\delta} & & M
\end{array}
\]

where

\[
\begin{array}{l}
\exists \\
\end{array}
\]

so that \( \prime L_\bullet(M) \) is a truncated resolution of \( M \). One obtains a canonical bijection between the set of the equivalence classes of Pic-categories preepingled of type \((M, N)\) and the set \( \text{H}^2(\text{Hom}(\prime L_\bullet(M), N)) \). The exactitude of the complex \( L(M) \) gives us the triviality of the classification of Pic-categories preepingled of type \((M, N)\) which are strict, i.e. all Pic-categories preepingled of type \((M, N)\) which are strict, are equivalent.

Finally chapter III gives us the construction of the solution of two universal problems: \textit{problem of making objects “unity objects”} and \textit{problem of reversing objects}.

Let be a \( \otimes\text{-AC category} \), \( \prime \) another \( \otimes\text{-AC category} \) whose base category is a groupoid, and \((T, \tilde{T}) : \xrightarrow{\prime} \text{a } \otimes\text{-AC functors} \). We try to make the objects \( T\prime A \) of \( A' \in \text{Ob}(\prime) \), “unity object”, i.e. we try to get:

1°) A \( \otimes\text{-ACU category } \mathcal{P} \)

2°) A \( \otimes\text{-AC functor } (\text{\( D, \tilde{D} \)) : \xrightarrow{\prime} \mathcal{P}} \)

3°) A \( \otimes\text{-isomorphism } \lambda : (\text{\( D, \tilde{D} \))} \circ \text{\( T, \tilde{T} \)) \xrightarrow{\sim} (\text{\( I_{\mathcal{P}}, \tilde{I}_{\mathcal{P}} \))} \)

where \((I_{\mathcal{P}}, \tilde{I}_{\mathcal{P}})\) is the \( \otimes\text{-constant functors } 1_{\mathcal{P}} \) from \( \prime \) to \( \mathcal{P} \). The triple \((\mathcal{P}, (D, \tilde{D}), \lambda)\) must be universal for triples \((E, \tilde{E}, \mu)\) satisfying 1°, 2°, 3°.

For the description of the triple \((\mathcal{P}, (D, \tilde{D}), \lambda)\), we introduce a quotient category of a \( \otimes\text{-AC category} \) as follows:

Let be a \( \otimes\text{-AC category} \), \( Y \) a \textit{multiplicative subset} of (that means a subset of the set of all endomorphisms of such that \( \text{Id}_X \in Y \) for all \( X \in \text{Ob}(\prime) \) and the tensor product of two arrows of \( Y \) belongs to \( Y \)). The \( \otimes\text{-AC category quotient } A^Y \) with respect to \( Y \) is the solution of the universal problem

\[
(K, \tilde{K}) : \xrightarrow{\prime}, \quad K(u) = \text{Id} \text{ for all } u \in Y
\]

where \( B \) is a \( \otimes\text{-AC category} \) and \((K, \tilde{K})\) a \( \otimes\text{-AC functor} \).

Now let us give an idea of the construction of the triple \((\mathcal{P}, (D, \tilde{D}, \lambda))\) for \( \prime \neq \emptyset \):
$1° \ \text{Ob}(\mathcal{P}) = \text{Ob}(\mathcal{I})$

$2° \ \text{Hom}_{\mathcal{P}}(A, B) = \varphi(A, B)_{R_{A,B}}, A, B \in \text{Ob}(\mathcal{P})$

$\varphi(A, B)$ being the set of all triples $(A', B', u)$ where $A', B' \in \text{Ob}(\mathcal{I})$, $u \in F(I)$, $u : A \otimes T A' \longrightarrow B \otimes T B'$; $R_{A,B}$ the equivalence relation defined in $\varphi(A, B)$ as follows

$$(A'_1, B'_1, u) R_{A,B} (A'_2, B'_2, u)$$

if and only if there are objects $C'_1, C'_2$ and isomorphisms

$$u' : A'_1 \otimes C'_1 \sim A'_2 \otimes C'_2, \quad v' : B'_1 \otimes C'_1 \sim B'_2 \otimes C'_2$$

of $\mathcal{I}$ such that the following diagram commutes in $\mathcal{I}$-AC quotient category of $\mathcal{I}$ with respect to the multiplicative subset of generated by the endomorphisms of the form $T(c_{A', A})$;

$\square$

We denote by $[A', B', u]$ the class which has $(A', B', u)$ as representative

$3° \ \text{Composition of arrows in } \mathcal{P}$. Let $[A', B, u] : A \longrightarrow B, [B'', C'', v] : B \longrightarrow C$ be arrows in $\mathcal{P}$. We define

$$[B'', C'', v] \circ [A', B', u] = [A' \otimes B'', B' \otimes C', w] : A \longrightarrow C$$

where $w$ is such that the following diagram commutes:

$\square$

$4° \ \otimes$-structure on $\mathcal{P}$

$$A \otimes E \ (\text{in } \mathcal{P}) = A \otimes E \ (\text{in } \mathcal{I})$$

$$[A', B', u] \otimes [E', F', v] = [A' \otimes E', B' \otimes F', w]$$

where $w$ is defined by the commutative diagram (1)

$5° \ \text{ACU constraint in } \mathcal{P}$.

$$([A', A', a \otimes \text{Id}], [A', A', c \otimes \text{Id}], (1_{\mathcal{P}} = TA_0, g_A = [A'_0 \otimes A', A', t_A], d_A = [A'_0 \otimes A', A', p_A]))$$

where $A'_0$ is a fixed object of $\mathcal{I}$, $A'$ an arbitrary object of $\mathcal{I}$, $g_A$ and $d_A$ natural isomorphisms

$$g_A : A \longrightarrow 1_{\mathcal{P}} \otimes A, \quad d_A : A \longrightarrow A \otimes 1_{\mathcal{P}}$$

with $t_A$ and $p_A$ defined by the commutativity diagrams (2)
\( (D, \tilde{D}) \) is defined by

\[
DA = A, \quad D_u = [A', A', \mu \otimes \text{Id}_{T'A'}], \quad \tilde{D}_{A,B} = \text{Id}_{A\otimes B}
\]

For the problem of reversing objects, let us consider a \( \otimes \)-category \( \mathcal{C} \) with a ACU constraint \((a, c, (1, g, d))\) a \( \otimes \)-category \( \mathcal{C}' \) with a ACU constraint \((a', c', (1', g', d'))\), the base category of which is a groupoid, and a \( \otimes \)-ACU functor \((F, \tilde{F}) : \mathcal{C}' \to \mathcal{C} \). We try to find a \( \otimes \)-ACU category \( \mathcal{P} \) and a \( \otimes \)-ACU functor \((D, \tilde{D}) : \mathcal{C} \to \mathcal{P} \) having the following properties

1. \( DFX' \) is invertible in \( \mathcal{P} \) for all \( X' \in \text{Ob}(\mathcal{C}') \)

2. For all \( \otimes \)-ACU functor \((E, \tilde{E})\) from \( \mathcal{C} \) to a \( \otimes \)-ACU category such that \( EFX' \) is invertible in \( \mathcal{P} \) for all \( X' \in \text{Ob}(\mathcal{C}') \), there exists a \( \otimes \)-ACU functor \((E', \tilde{E}')\), unique up to \( \otimes \)-isomorphism, from \( \mathcal{P} \) to \( \mathcal{P} \) such that \( (E, \tilde{E}) \simeq (E', \tilde{E}' \circ (D, \tilde{D})) \).

This problem is reduced by the first by putting \( ' = \mathcal{C}' = \mathcal{C} \times \mathcal{C}' \), \( TX' = (FX', X') \) and by remarking that if \( \mathcal{C} \), \( \mathcal{C}' \), are \( \otimes \)-ACU categories, \( \otimes \text{ACU}(\mathcal{C}, \mathcal{C}') \) the category of all \( \otimes \)-ACU functors from \( \mathcal{C} \) to \( \mathcal{C}' \), then there is a canonical equivalence of categories

\[
\text{ACU}(\mathcal{C} \times \mathcal{C}', \mathcal{C}) \to \text{ACU}(\mathcal{C}, \mathcal{C}') \times \text{ACU}(\mathcal{C}', \mathcal{C})
\]

The \( \otimes \)-ACU category \( \mathcal{P} \) thus defined is called the \( \otimes \)-category of fractions of the category \( \mathcal{C} \) with respect to \((\mathcal{C}', (F, \tilde{F}))\). The \( \otimes \)-category of fractions of \( \mathcal{C}^{\text{is}} \) with respect to \((\mathcal{C}^{\text{is}}, (\text{Id}_{\mathcal{C}^{\text{is}}}, \text{Id}))\) is a Pic-category which is called the Pic-envelope of the category \( \mathcal{C} \), and denoted by Pic(\( \mathcal{C} \)).

For an application of the Pic-envelope, we take \( \mathcal{C} = P(R) \), category of all finitely generated \( R \)-modules (\( R \) a ring) and \( \mathcal{P} = \text{Pic}(P(R)) \), then one obtain

\[
\pi_0(\mathcal{P}) \simeq K^0(R)
\]

\[
\pi_1(\mathcal{P}) \simeq K^1(R)
\]

where \( K^0(R) \) is the Grothendieck group and \( K^1(R) \) the whitehead group [1].
The use of the $\otimes$-category of fractions of a $\otimes$-ACU category gives us the following result:

Let $\mathcal{C}$ be a $\otimes$-ACU category, $Z$ an arbitrary object of $\mathcal{C}$ different from the unity object $1$, $S$ the functor from $\mathcal{C}$ to $\mathcal{C}$ defined by

$$X \mapsto X \otimes Z.$$ 

The suspension category of the $\otimes$-ACU category $\mathcal{C}$ defined by the object $Z$ is the triple $(\mathcal{P}, i, p)$ which solves the universal problem for triples $(j, q)$ where $\mathcal{I}$ is a category, $j$ a functor from $\mathcal{C}$ to, and $q$ an equivalence of categories from to, so that the following diagram commutes

$$[]$$

up to natural isomorphism. In the case where $\mathcal{C}$ is the homotopy category of pointed topological spaces, together with the smash $\Wedge$ (the smash $\Wedge$ of two spaces $X$ and $Y$, with the base points $x_0$ and $y_0$, is obtained from the product $X \times Y$ by $\Wedge$ the subset $\{x_0, y_0\}$ to a single point which is taken as the base point of $\mathcal{C}$, and the usual ACU constraint; and $Z$ is the 1-sphere $S^1$ hence $S^1$ is the suspension functor, we get the well-known definition of the suspension category.

Let $\mathcal{C}'$ be the $\otimes$-stable subcategory of $\mathcal{C}$ generated by $Z$ and $\mathcal{P}$ the $\otimes$-category of fractions of $\mathcal{C}'$ with respect to $(\mathcal{C}', (F, \text{Id}))$ where $F : \mathcal{C}' \longrightarrow \mathcal{C}$ is the inclusion functor. One obtains a functor $G : \mathcal{P} \longrightarrow \mathcal{P}$ from the suspension category to the $\otimes$-category of fractions of $\mathcal{P}$. If $G$ is not faithful, that is the case of the homotopy category of pointed topological spaces, together with the smash $\Wedge$ and the 1-sphere $S^1$; then it is impossible to construct in $\mathcal{P}$ a law $\otimes$ such that $\mathcal{P}$ together with this law is a $\otimes$-ACU category, $iZ$ invertible in $\mathcal{P}$, and $i$ embedded in a pair $(i, \tilde{i})$ which is a $\otimes$-ACU functor from $\mathcal{C}$ to $\mathcal{P}$. 

8
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