Crystals and the De Rham cohomology of Schemes[1]

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(Notes by I. Coates and O. Jussila)

Introduction

These notes are a rough summary of five talks given at I.H.E.S in November and December 1966. The purpose of these talks was to outline a possible definition of a p-adic cohomology theory, via a generalization of the De Rham cohomology which was suggested by work of Monsky-Washnitzer [?] and Manin [?].

The contents of the notes are by no means intended to be a complete theory. Rather, they outline the start of a program of work which has still not been carried out[2].

1. De Rham cohomology

1.1. Differentiable Manifolds. Let X be a differentiable manifold, and $\underline{\Omega}^{\bullet}_{X/\mathbb{C}}$ the complex of sheaves of differential forms on X, whose coefficients are complex valued differentiable functions on X.

[1]This text has been transcribed by Mateo Carmona
https://agrothendieck.github.io/

[2]For a more detailed exposition and progress in this direction, we refer to the work of P. Berthelot, to be developed presumably in SGA 8.

Theorem **1.1.** (De Rham) — *There is a canonical isomorphism*

$$\mathrm{H}^*(X,\mathbf{C}) \xrightarrow{\sim} \mathrm{H}^*(\Gamma(X,\underline{\Omega}^\bullet_{X/\mathbf{C}})),$$

where $\mathrm{H}^*(X,\mathbf{C})$ *is the canonical cohomology of* X *with complex coefficients.*

To prove this, one observes that, by Poincaré's lemma, the complex $\underline{\Omega}^\bullet_{X/\mathbf{C}}$ is a *resolution* of the constant sheaf $\underline{C}$ on X, and that the sheaves $\underline{\Omega}^j_{X/\mathbf{C}}$ are *fine* for $J \geq 0$, so that $\mathrm{H}^i(X,\underline{\Omega}^j_{X/\mathbf{C}}) = 0$ for $i > 0$ and $j \geq 0$, whence the assertion.

An analogous result holds for the complex of sheaves of differential forms on X, whose coefficients are real valued differentiable functions on X.

1.2.

1.3.

1.4.

1.5.

1.6. Criticism of the ℓ-adic cohomology. If X is a scheme of finite type over an algebraically closed field k, and ℓ is any prime number *distinct*[3] from the characteristic of k, the ℓ-adic cohomology of X is defined to be

1.7.

1.8. Proposals for a p-adic Cohomology. We only mention two proposals, namely Monsky and Washnitzer's method via special affine liftings (which we discuss in n° 2), and the method using the fppf (faithfully flat and finite presentation) topology.

By analogy with the ℓ-adic cohomology, the essential idea of the fppf topology was to consider the cohomology of X/k, with respect to the fppf topology, with coefficient groups in the category C^ν of finite schemes of $\mathbf{Z}/p^\nu\mathbf{Z}$-modules. Examples of such schemes of modules are

2. The cohomology of Monsky and Wishnitzer

2.1. Approach via liftings.

Suppose X_0 is a scheme on a perfect field k

[3] the ℓ-adic cohomology is still defined for ℓ equal to the characteristic of k, but it no longer has too many reasonable properties.

3. Connections on the De Rham cohomology

For the definition of a *connection* and a *stratification* on a sheaf, see Appendix I of these notes.

4. The infinitesimal topos and stratifying topos

We now turn to the definition of a more general category of coefficients for the De Rham cohomology. To this end we introduce two ringed topos, the *infinitesimal topos* and the *stratifying topos*.

We shall see later that in fact these two topos work well only in characteristic 0

5. Cěch calculations

We now consider the cohomology of the infinitesimal topos and the stratifying topos[4]

6. Comparison of the Infinitesimal and De Rham Cohomologies

6.1. The basic idea. Let X be a scheme above S, and F a quasi-coherent Module on X fortified with a stratification relative to S.

7. The crystalline topos and connecting topos

7.1. Inadequacy of infinitesimal topos. Let X_0 be a scheme above a perfect field k of characteristic $p > 0$. Then, regarding X_0 as being above $S = \operatorname{Spec} W(k)$ instead of k, the infinitesimal cohomology

$$\mathrm{H}^*((X_0/S)_{\mathrm{inf}}, \underline{O}X_0)$$

is a graded module

[4]For a general discussion of the cohomology of a topos, see (SGA 4 V).

Appendix

Let X be a scheme above the base S, and F a Module on X. For each positive integer n,

REFERENCES

[1] ARTIN, M., GROTHENDIECK, A., J.L. VERDIER — *Cohomology étale des schémas*, Sém. Géom. Alg. IHES, 1963-64 (SGA 4), à paraître dans North Holland Pub. Cie.